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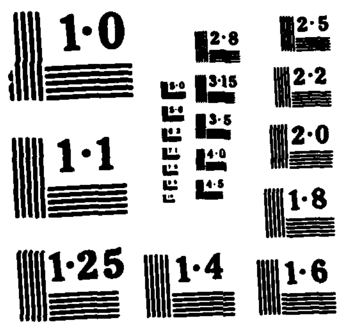

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PROBABILISTIC ANALYSIS OF NONLINEAR STRUCTURES SUBJECTED TO TRANSIENT LOADS

Part 1 of 2

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Final Report

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A technique for the analysis of reliability in linear and nonlinear systems that can be modeled as single-degree-of-freedom (SDF) structures is investigated and established. Transition probabilities are established using conditional probability and problems for stationary and nonstationary input are shown. Methods for computational enhancements are investigated. Multiple point failure analysis of multiple-degree-of-freedom (MDF) systems is addressed separately. Peak responses are expanded in a series involving the input and systems parameters and moments of the peak response are determined. Margins of survival are established using random failure levels and the probability of survival is computed. A second method using the equations of a numerical technique to form the joint probability density of the response measures is also presented.

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SUMMARY

Procedures for evaluating the probability of failure of structures subjected to blast and shock loadings have been established. For simple systems that can be represented by a single-degree-of-freedom, a procedure is used that computes the transition probabilities between states of the system using conditional probability over calculational time intervals. The system and loading parameters that are random must be adequately characterized and the response space must be discretized in a reasonable way to represent various states of response. The result is a representation of the distribution of the response within the discretized response space. All other probability outside the discretized response space is lost so that the first passage probability outside the selected bounds of response can be obtained. Various schemes to enhance this procedure computationally have been demonstrated. The procedure is not limited to elastic systems or to stationary input, and measures of the response other than displacement may be considered. The first passage of dissipated energy in an elasto-plastic system was specifically addressed and the probability of failure given a random failure level has been computed.

For more complex systems that require multiple-degree-of-freedom representation, a procedure is presented that expands the peak response in a Taylor series in terms of the loading and system parameters. The peak response is determined by conventional deterministic methods. The mean and variance of the response at each location are determined by numerically approximating partial difference terms. Given the mean and variance of the failure criteria, a margin of survival can be computed. When the form of the

distribution of the margin of survival is specified, then the probability of survival and its complement, and the probability of failure can be numerically determined by integration of a multivariate distribution function.

An additional approach for complex systems has been demonstrated that uses the relationships between parameters in a numerical sequence. The Markov property allows the random character of the response measures and their correlations to be computed at each time step. The result is the evolution of a stochastic process with time.

PREFACE

Section II, "Stochastic Analysis of Mechanical Systems," was funded under an Air Force Office of Scientific Research (AFOSR) grant, program element 61101F, and has been published elsewhere as part of AFWL-TR-82-123. Sections III and IV were funded under program elements 62601F and 64711F respectively. This technology is being transitioned to field problems concerned with random loads and random material properties.

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ILLUSTRATIONS (Concluded)

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I. INTRODUCTION

1. MOTIVATION AND OBJECTIVES

Structures are designed for a variety of purposes. Some function as weather shelters (conventional buildings) while others protect their occupants from more severe outside environments like blast and shock (protective structures). Some protect the outside environment from their contents (containment structures). The purposes of a structure are usually translated into performance criteria. The goal of the structural engineer is to design, analyze, and construct structures that satisfy the performance criteria. Since the inputs, the material, and the structural performance are not strictly deterministic, the engineer must be concerned with the probability that the structure will respond to all applied inputs within the limits set by the criteria. This probability is called reliability. When a structure's response is designed to be far into the inelastic regime, as in the case of most protective structures, the consideration of reliability becomes both more important and more difficult. The difficulty arises from the requirement to predict the probabilistic nature of the nonlinear response.

The reliability of a structure should be determined in any comprehensive analysis since, in practice, most inputs and structures are random. This is especially true for protective structures in a blast and shock environment. Dynamic shock inputs are often considered random because (1) the energy expended at the shock source cannot be directly measured, and/or (2) the physical properties of the medium connecting the shock source to the point of interest cannot be observed. Structural systems are considered random because

(1) the parameters in the constitutive laws governing behavior of the structural materials are random, and/or (2) the structural geometry does not precisely match the intended configuration. (There is another type of uncertainty that has been mistaken for randomness and that is modeling accuracy. Model inadequacy is a systematic error and will not be directly addressed in this study.)

The purpose of a reliability analysis is to compute the probability of survival of a structure. Reliability computation is difficult when the structural response is nonlinear and has large deflections. Therefore, one of the ultimate goals in reliability analysis is to specify a technique for the computation of reliability of structures executing nonlinear, large deflection response.

The purpose of this investigation is to establish techniques for the analysis of reliability of dynamically loaded structures that behave in a highly nonlinear way. A progression of problem complexity will cover a range of techniques from single-degree-of-freedom (SDF) systems subjected to stationary inputs to multiple-degree-of-freedom (MDF) systems subjected to transient loads. There are four requirements to these approaches. First, the random character of the structural system must be identified. Second, the random character of the loads must be described. Third, the random character of the failure criteria must be established. Finally, the fourth requirement is to establish a method using the first three requirements to produce a prediction of probability of failure of a structural system. Necessarily, this fourth step uses an intermediate step of determining the probabilistic nature of the response and then applies the random failure criteria to establish the probability of failure or its complement reliability.

2. LITERATURE REVIEW

There are several areas of research interest which have a direct bearing on the topic under consideration, damage analysis of randomly excited systems with random characteristics. The purpose of this literature review is to summarize the results of some recent studies which relate to the present topic. This literature review deals with investigations in the probabilistic theory of structural dynamics, and investigations that characterize the appropriate loading and system parameters using probabilistic models. Specifically, first, some texts and papers discussing the characterization of dynamic random structural response will be listed. Next, specific studies useful in the analysis of failure due to peak response are given. The characterization of damage is important for structures which can fail due to damage accumulation; therefore, papers in this area are discussed next. Then the studies which take advantage of the Markov property of a structural response and its measures will be considered. Finally, some papers which discuss the random character of environments and structural systems are given.

There are several texts which deal with many aspects of the probabilistic theory of structural dynamics. Among these are the books by Lin (Ref. 1), Crandall and Mark (Ref. 2), Newland (Ref. 3), Crandall (Refs. 4 and 5), and Clough and Penzien (Ref. 6). These books treat, in detail, the problem of computation of response moments for structural systems. In particular, these books present techniques for finding the moments of the structural response of linear SDF, MDF, and continuous systems, excited by stationary and nonstationary random inputs. The systems considered all have deterministic, constant parameters. Some relatively simple, nonlinear problems are also considered.

For example, some approximate means for computing the response moments of nonlinear, elastic, SDF structures are given in Reference 1. These texts also consider such problems as the first passage problem for linear SDF systems, the fatigue problem, and the Markov character of the response of systems excited by white noise.

Many general papers broadly characterizing one or more aspects of structural response to a random input have been written. For example, the paper by Rice (Ref. 7) treats the response moments of linear and simple nonlinear systems. It also characterizes the frequency of passage of a response random process beyond a barrier of fixed height. The paper by Ang (Ref. 8) gives a method for computing the response moments of a linear system, and then shows how to use these for first passage and other computations. The papers by Wirsching and Yao (Refs. 9 and 10) and Bogdanoff, Goldberg, and Bernard (Ref. 11) evaluate the response of linear structures to earthquake-type disturbances and assess structural safety in light of the response. Uhlenbeck and Ornstein (Ref. 12) and Wang and Uhlenbeck (Ref. 13) consider the Markov character of the response of linear systems to white noise input and use this to derive the Fokker Planck equations for these systems. They then solve the Fokker Planck equations to obtain the transition probabilities for the structural response. Caughey (Ref. 14) derives and solves the Fokker Planck equation for nonlinear elastic systems and obtains the response moments and transition probabilities for the structural response. Goldberg, Bogdanoff, and Sharpe (Ref. 15) and Toland and Yang (Ref. 16) analyze the response of simple, nonlinear, elastic structures and evaluate response moments and first passage probabilities. Vanmarcke, Yanev, and De Estrada (Ref. 17), Paez and Yao (Ref. 18), and Iyengar and Iyengar (Ref. 19) characterize the response of

SDF hysteretic structures. They compute (1) moments and first passage probabilities for the displacement response, and (2) the moment of accumulated plastic strain and permanent set in the structures. Wen (Ref. 20) specifies a means for analyzing MDF hysteretic structures. He places each hysteretic spring with an equivalent, higher order, linear spring. Response moments are then analyzed.

Investigators interested in identifying the probability of failure of a structure often consider the first passage problem, or equivalent peak response problem. Many of these investigations consider SDF systems. For example, the papers by Yang (Refs. 21 and 22), Yang and Shinozuka (Ref. 23), Roberts (Refs. 24 and 25), Lin (Ref. 26), and Corotis, Vanmarcke, and Cornell (Ref. 27) use various techniques to find the first passage probability for linear, SDF oscillators. Paez (Ref. 28) specifies a technique for computing the peak response probability distribution for an SDF system. In a general paper, Ang (Ref. 8) defines an approximate means for computing the first passage probability for an MDF system. Koopmans, Qualls, and Yao (Ref. 29) and Kojwithya (Ref. 30) specify bounds on the peak response probability distribution for linear, MDF systems.

In assessing the chance of failure of a structure, the potential for accumulation of damage must be considered. Several authors have suggested codes and formulas for classifying the damage in a structure. For example, qualitative measures of structural damage are discussed in papers by Whitman et al. (Refs. 31 and 32), Housner and Jennings (Ref. 33), Hart (Ref. 34), and Hsu (Ref. 35). Quantitative measures of the damage accumulated in simple structural members are also available. For example, the papers by Yao and Munse (Ref. 36), Tang and Yao (Ref. 37), Oliveira (Ref. 38), and Kasiraj and

Yao (Ref. 39) suggest quantitative measures of accumulated structural damage. Some of these measures of damage are based on Palmgren-Miner-type damage accumulation laws. Some literature surveys on the subjects of damage assessment and measures of damage have been written by Yao (Refs. 40 and 41).

In investigations into the probabilistic structural response of mechanical systems, researchers have sometimes taken advantage of the Markov character of the response to white noise and filtered white noise inputs. Notably, in studies of the first passage probability for linear system response, Yang and Shinozuka (Refs. 42 and 43), Rosenblueth and Bustamante (Ref. 44), Gray (Ref. 45), Crandall, Chandiramani, and Cooke (Ref. 46), and Paez and Yao (Ref. 47) have used the Markov character of the response. All these studies concern linear, SDF structures. In addition, Bogdanoff (Refs. 48 and 49), Bogdanoff and Krieger (Ref. 50), and Paez, Tang, and Yao (Ref. 51) have shown how a Markov chain approach can be used in the failure analysis of structural systems. When the damage transition probability matrix is provided, and when the probability distribution of the failure-causing damage level is known, their approach can be used to find the probability of failure of a structural system.

The literature review in this section summarizes investigations into various aspects of the probabilistic structural response problem. Many papers dealing with the probabilistic theory of linear structural response are available, and some of these are mentioned in review. Fewer papers analyzing the peak response of linear structures have been written, and even fewer deal with nonlinear structures. Also, only a few papers dealing with the accumulation of damage in randomly excited structures are available, though much effort is being devoted to the damage accumulation problem at this time.

Damage accumulation models may also be affected by the development of more sophisticated constitutive models. There is extensive work being done in this area of constitutive modeling of concrete soils. Generally the failure criteria will be more adequately specified as material models are developed that are more accurate in failure regimes. None of the probabilistic papers reviewed considers the large displacement response of structural systems. A few of the papers in the literature take advantage of the Markov character of structural response; most of these consider the linear response behavior of linear systems. This property of structural response is important in the development of this investigation and can be extended to systems that respond in a nonlinear way.

The random character of the environment for pwotective construction has been addressed in some detail, but the information is not available in the open literature. Finally, there are some papers that report the specific random character of some of the environment and structural system parameters. For example, Reference 52 summarizes recent studies in the strength of reinforced concrete.

II. STOCHASTIC ANALYSIS OF SIMPLE MECHANICAL SYSTEMS

1. INTRODUCTION

In this section, some SDF systems are analyzed using conditional probability approaches. Linear elastic, nonlinear elastic, and hysteretic SDF systems are analyzed. The parameters of most of the systems considered are deterministic constants; however, one of the systems analyzed is assumed to have a random capacity for damage accumulation. The input used to excite most of the systems considered here is a band-limited white noise random process. In one instance, however, a nonstationary random process is used to excite the system under consideration.

Several types of probabilistic results are obtained for the SDF systems analyzed in this section. First passage probabilities for some measures of SDF system response are computed. Finally, the failure probability is computed for a system assumed to degrade following a particular damage law.

2. FIRST PASSAGE OF LINEAR SDF SYSTEMS

a. Theoretical analysis--In this section a class of first passage problems for SDF systems is solved. In order to solve the first passage problem, the probability that some measure of system response will pass outside a preestablished barrier at, or before, a specific time will be computed. When the response level corresponding to failure in an SDF system can be defined in terms of one response measure and when it is deterministic and known, the probability of failure of an SDF system can be computed using a first passage analysis.

The only requirements placed upon the input in the following analysis are: (1) that it be accurately characterized in discrete time, and (2) that the values of the input random process at consecutive time points be independent. A temporarily stationary input random process implies that the input is a band-limited white noise. In the analysis of SDF systems, the assumption of a white noise input is not severe. This is true since, in general, only that value of the spectral density of the input near the natural frequency of the system under consideration is important in its influence on the structural response.

First passage analyses can be executed using a Markov chain framework. The only quantities that must be generated are the transition probabilities. One means for generating transition probabilities will be presented in this section.

The equation of motion governing the response of a linear, base-excited SDF system is

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = -\ddot{x} \tag{1}$$

where y is the relative displacement response of the SDF system, $\ddot{x}$ is the base acceleration, ω_n is the system natural frequency, ζ is the damping factor, and dots denote differentiation with respect to time.

In order to obtain the transition probabilities required for the probabilistic analysis, this equation is first discretized in time using a central finite difference approximation. The resulting approximation to the governing equation is

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{\Delta t^2} + 2\zeta\omega_n \frac{y_{j+1} - y_{j-1}}{2\Delta t} + \omega_n^2 y_j = \ddot{x}_j, \qquad j = 0,\ldots,N \tag{2}$$

Here the equation of motion is written for time $t_j = j\Delta t$, $j = 0,\ldots,N$; Δt is the discrete time interval for the problem. Equation 2 can be solved for y_{j+1} and the result is

$$y_{j+1} = (1 + \zeta\omega_n \Delta t)^{-1} [\Delta t^2 \ddot{x}_j + (\Delta t^2 \omega_n^2 - 2) y_j + (1 - \zeta\omega_n \Delta t) y_{j-1}] ,$$

$$j = 0,\ldots,N \qquad (3)$$

When the system parameters are constants and input excitation is a stochastic process, the SDF system response is also a stochastic process. In this case the system response is still governed by Equation 3. When the actual value of the input at time t_j is specified and the response values at t_{j-1} and t_j are known, the the response value at t_{j+1} can be approximately computed. In the following it has been assumed that the system parameters are constants and the input excitation is a stochastic process. It is assumed that the random variables composing the stochastic process are independent at consecutive times, and that the probability distribution of the input is known at each time t_j, $j = 0,\ldots,N$. To be consistent with standard random process notation, the symbols denoting response and input in Equation 3 are capitalized when they represent stochastic processes. The input stochastic process is denoted X_j, $j = 0,\ldots, N$; the response stochastic process is Y_j, $j = 0,\ldots,N$.

The transitional probabilities governing the SDF structure response will be obtained using the definition of conditional probability and Equation 3. The probability that the structural response falls in the interval (y_a, y_b) at time t_{j+1}, given that the response at time t_{j-1} is y_{k_0} and the response at time t_j is y_{k_1}, is given by

$$P\left(y_a < Y_{j+1} \le y_b \middle| Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\right)$$

$$= P\Big(y_a < (1 + \zeta\omega_n \Delta t)^{-1} [\Delta t^2 \ddot{X}_j + (\Delta t^2 \omega_n^2 - 2) Y_j + (1 - \zeta\omega_n \Delta t) Y_{j-1}]$$

$$\le y_b \Big| Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\Big), \qquad j = 0,\ldots,N \qquad (4)$$

The expression on the left is simply the conditional probability described above. The expression on the right simply uses the definition for Y_{j+1}, established in Equation 3, in place of y_{j+1}.

Since specific values for the response at times t_{j-1} and t_j are given in Equation 4, these values can be used inside the inequality on the right-hand side of Equation 4. Moreover, the inequality can be solved for the input random variable, $\ddot{X}_j$. The resulting expression is

$$P\left(y_a < Y_{j+1} \leq y_b \middle| Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\right)$$

$$= P\left(\frac{y_a - 2y_{k_1} + y_{k_0}}{\Delta t^2} + 2\zeta\omega_n \frac{y_a - y_{k_0}}{2\Delta t} + \omega_n^2 y_{k_1} < \ddot{X}_j \leq \right.$$

$$\left. \frac{y_b - 2y_{k_1} + y_{k_0}}{\Delta t^2} + 2\zeta\omega_n \frac{y_b - y_{k_0}}{2\Delta t} + \omega_n^2 y_{k_1} \middle| Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\right), \quad j = 0,\ldots,N \tag{5}$$

When specific values for y_a, y_b, y_{k_0} and y_{k_1} are provided, along with values for the system parameters, then a numerical value can be evaluated for the probability in Equation 5. This is true since the probability distribution of $\ddot{X}_j$ has been assumed known.

To establish the framework for the computation of an entire collection of probabilities characterizing the structural response, the present problem is now discretized in response space. Let $-c_a$ and c_b represent the lower and upper first passage barriers. The first passage probability is the probability that Y_j, $j = 0,\ldots N$ assumes a value outside the interval $(-c_a, c_b)$ at or before the time t_j. The displacements are discretized into M equally spaced values in the range $(-c_a, c_b)$. The displacements are

$$y_k = (k - 1/2)\,\Delta y - c_a\,, \qquad k = 1,\ldots,M \tag{6}$$

where

$$\Delta y = (c_a + c_b)/M \tag{6a}$$

Now the probability that the response at t_{j+1} assumes a value in an interval of width Δy surrounding the displacement y_k can be computed by using $y_k - \Delta y/2$ in place of y_a and $y_k + \Delta y/2$ in place of y_b in Equation 5. When the response at t_{j+1} is assumed to be in $(y_k - \Delta y/2,\ y_k + \Delta y/2]$, then the response equals y_k. This assumption makes the random variables in the stochastic process, Y_j, $j = 0,\ldots,N$, discrete valued. The assumption results in an accurate representation of the response when Δy is small enough. Mathematically, the assumption described above is written

$$P\left(Y_{j+1} = y_{k_2} \middle| Y_{j-1} = y_{k_0},\ Y_j = y_{k_1}\right)$$

$$\simeq P\left(y_{k_2} - \Delta y/2 < Y_{j+1} < y_{k_2} + \Delta y/2 \middle| Y_{j-1} = y_{k_0},\ Y_j = y_{k_1}\right) \quad \begin{array}{l} j = 0,\ldots,N \\ k_0,k_1,k_2 = 1,\ldots,M \end{array} \tag{7}$$

The numerical values of all the transition probabilities represented by Equation 7 can be computed. In all, there are M^3 values. These probabilities characterize the transitions of the response from points within the first passage bounds at times t_{j-1} and t_j to points within the first passage bounds at time t_{j+1}. These probabilities do not represent all the possible displacement versus time paths that the SDF system might execute during response to a random input. Only those paths remaining within the interval $(-c_a,\ c_b)$ are represented. Those transition probabilities which correspond to paths which originate or pass outside the first passage barriers will not be used in the present analysis. To completely characterize its probabilistic response, starting probabilities for the SDF system under consideration must be specified. These starting probabilities are the values of the joint probability mass function (pmf) of the random variables Y_{-1} and Y_0. Y_0 is the first random variable in the response stochastic process; Y_{-1} is a random variable defined at the arificial time $t_{-1} = -\Delta t$. Y_{-1} is defined solely to facilitate characterization of the probability distribution of the starting velocity. The joint pmf of Y_{-1} and Y_0 defines

$$p_{Y_{-1}Y_0}\left(y_{k_0}, y_{k_1}\right) = P\left(Y_{-1} = y_{k_0}, Y_0 = y_{k_1}\right), \qquad k_0, k_1 = 1, \ldots, M \tag{8}$$

Generally, the system considered will have a zero start condition. This means that the system starts with zero displacement and zero velocity. This zero start condition is characterized by the joint pmf

$$p_{Y_{-1}Y_0}\left(y_{k_0}, y_{k_1}\right) = \begin{cases} 1, & y_{k_0} = y_{k_1} = 0 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

In situations where none of the discrete displacements actually equals zero, the joint pmf is set to one at the argument (y_{k_0}, y_{k_1}) where $y_{k_0} = y_{k_1}$ is the realization of the random variable which is smallest in absolute value.

With the information provided in Equations 5, 7, and 8 or 9, the probability mass function of the response stochastic process can be propagated through time. First using Equations 7 and 8 the joint pmf of Y_0 and Y_1 can be obtained by recognizing that multiplying Equation 7 by Equation 8 will yield the joint pmf of Y_{-1}, Y_0, and Y_1. Summing over all k_0 will eliminate dependence Y_{-1} and yield the joint pmf of Y_0 and Y_1. Then this result is used with Equation 7 and the same process to get the joint pmf of Y_1 and Y_2, etc. The result is, at time t_{j-1} and t_j, we have the joint pmf of Y_{j-1} and Y_j. But recall that the transition probabilities, developed above, consider only those paths remaining in the spatial interval $(-c_a, c_b)$. So, in fact, the quantity

$P^*(Y_{j-1} = y_{k_0}, Y_j = y_{k_1})$, obtained in the manner described above, defines the triple joint probability that (1) the response at time t_{j-1} equals y_{k_0}, and (2) the response at t_j equals y_{k_1}, and (3) the SDF system response has not passed outside the interval $(-c_a, c_b)$ at or before the time t_j. (The star (*) superscript is included on $P^*(Y_{j-1}, Y_j = y_{k_1})$ to show that only a portion of a complete joint pmf is defined by the expression. The probabilistic description of paths passing outside the barrier is not obtained.)

Summing this expression over all k_0 yields

$$P^*\left(Y_j = y_{k_1}\right) = \sum_{k_0=1}^{M} P^*\left(Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\right), \qquad \begin{array}{l} j = 0,\ldots,N \\ k_1 = 1,\ldots,M \end{array} \tag{10}$$

This is the joint probability that the response at t_j equals y_{k_1} and the system response has not passed outside the interval $(-c_a, c_b)$ at or before t_j.

When Equation 10 is summed over all k_1, the "no passage" probability is obtained. If T_1 is defined as the random variable denoting the time when first passage of the SDF system relative displacement response outside the interval $(-c_a, c_b)$ occurs, then

$$P(T_1 > t_j) = \sum_{k_1=1}^{M} P^*\left(Y_j = y_{k_1}\right), \qquad j = 0,\ldots,N \tag{11}$$

This is the chance that no passage of the structural response outside the spatial interval $(-c_a, c_b)$ occurs in the time interval $(0, t_j]$. The first passage event is complementary to the event $T_1 > t_j$;

$$P(T_1 \leq t_j) = 1 - P(T_1 > t_j) , \qquad j = 0,\ldots,N \tag{12}$$

This is the first passage probability function, the desired quantity.

Note that, in order to use the present equations for the failure analysis of a structure, the structure must behave linearly (or nearly linearly) up to the failure point, and it must be assumed that failure occurs due to peak response. Moreover, the system must be accurately modeled as an SDF system.

On the other hand, the input can be modeled quite accurately. The input can be modeled as a nonstationary random process since the transition probabilities defined in Equation 5 depend on the distribution of x_i, and this distribution may vary with j.

The analysis described above is limited to the consideration of first passage problems where the barriers are constant and define a spatial interval $(-c_a, c_b)$. In fact, the analysis can be generalized to make the time domain discretization vary with j. One could simply redefine the barriers at every time t_j and derive the discrete displacements as in Equations 6 and 6a. This generalization would permit the performance of a probabilistic analysis of failure in systems where the strength properties degrade.

Since only those paths remaining between the first passage barriers are considered in the present analysis, only a portion of the probabilistic structural response is characterized. In fact, the entire spectrum of potential responses could be accounted for by inclusion in the analysis of two extra discrete displacements; one of these would be above the upper limit, c_b, and the other would be below the lower limit, $-c_a$. All paths passing outside the first passage barriers at a time t_j would be lumped in these two discrete displacements. The probability that a path originates outside a barrier and then returns to a displacement value inside the barriers would be set to zero. Using this approach guarantees that the probability mass function for displacement at a time t_j would have elements which sum to unity. The probability that the system response occupies the displacement value above the upper limit, c_b, or the displacement value below the lower limit, $-c_a$, at time t_j, would be the first passage probability.

Finally, note that the probability distribution of the stochastic process input has been left indefinite in Equation 5. In many situations the input will be defined as a normally distributed stochastic process, and in such cases Equation 5 will be specialized to

$$P\left(y_a < Y_{j+1} \leq y_b \middle| Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\right)$$

$$= \Phi\left(\sigma_j^{-1}\left(\frac{y_b - 2y_{k_1} + y_{k_0}}{\Delta t^2} + 2\zeta\omega_n \frac{y_b - y_{k_0}}{2\Delta t} + \omega_n^2 y_{k_1} - \mu_j\right)\right)$$

$$- \Phi\left(\sigma_j^{-1}\left(\frac{y_a - 2y_{k_1} + y_{k_0}}{\Delta t^2} + 2\zeta\omega_n \frac{y_a - y_{k_0}}{2\Delta t} + \omega_n^2 y_{k_1} - \mu_j\right)\right), \quad j = 0,\ldots,N \tag{13}$$

$\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable; μ_j and σ_j denote the mean and standard deviation of the input random process at time t_j.

b. <u>Numerical examples</u>--A computer program which executes the analyses developed in the previous section was written. A listing of the computer program is included with this report in Appendix B. Some numerical examples demonstrating the use of the computer program are summarized in this section.

The computer program computes the first passage probability function for a base-excited, linear SDF system. The input excitation is a stochastic process composed of a sequence of independent normal random variables. Each input random variable has mean zero and the standard deviation can decay exponentially with time. Zero start initial conditions for the SDF system are assumed.

One important feature of the program is the amount of storage required for the discrete functions used in the analysis. When an M-division spatial discretization is used in the problem solution, M^3 storage locations are required to hold the transition probabilities of Equation 7. In addition, M^3 storage locations are required to hold the products between the joint probabilities and the transition probabilities before the summation is executed. (Improvements in the analytical approach which diminish the required amount of storage are described later.) Other functions increase the storage requirements as functions of M and M^2; therefore, more than $2M^3$ storage locations are needed for this program. Clearly, this limits the range of discretizations that can be used in problem solutions.

The program listed in the Appendix was actually written to solve time and space normalized versions of the problem discussed in the previous section. In particular, the program performs a natural frequency independent computation, where the probability of first passage at or before a number of cycles of SDF system response is determined. In space, the upper and lower first passage bounds are written as a multiple of the root mean square (rms) value of the stationary response of an SDF system when the input is stationary.

The first problem to be summarized is the first passage of a base excited, linear SDF system. The input excitation is a stationary random process. Specifically, it is a band-limited white noise. The standard deviation of the input is listed in Table 1. The system and computation parameters are listed in Table 2. The first passage probability was determined at each computation cycle and the results are shown in Figure 1. (Note that curves relating to two examples are given in Figures 1 and 2. The

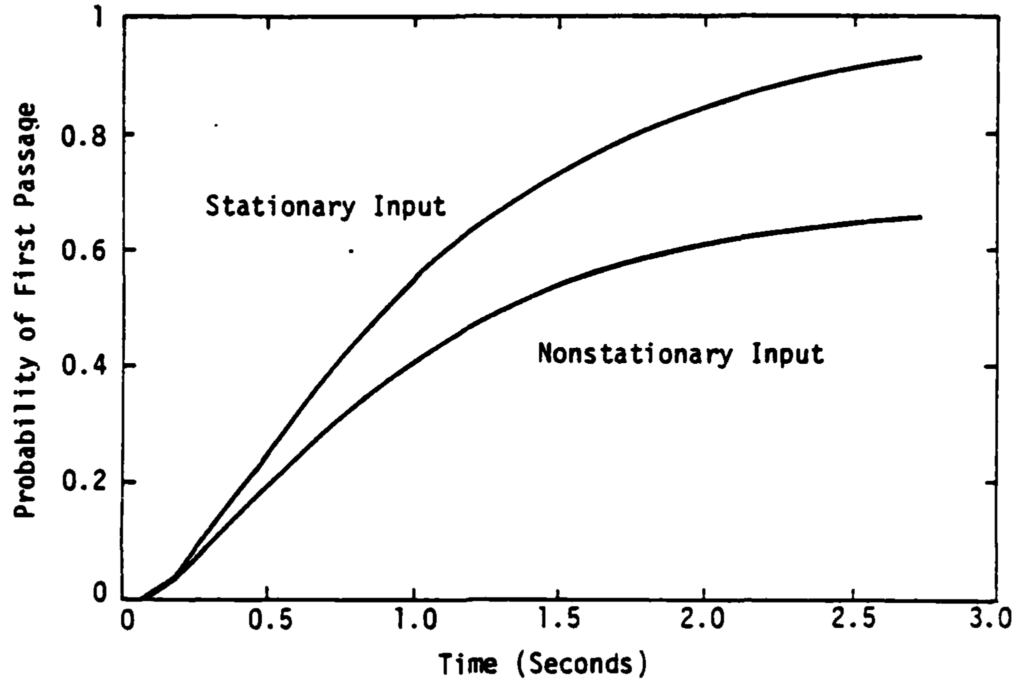

Figure 1. First passage probability distribution--stationary input, nonstationary input $\omega_n = 2\pi$.

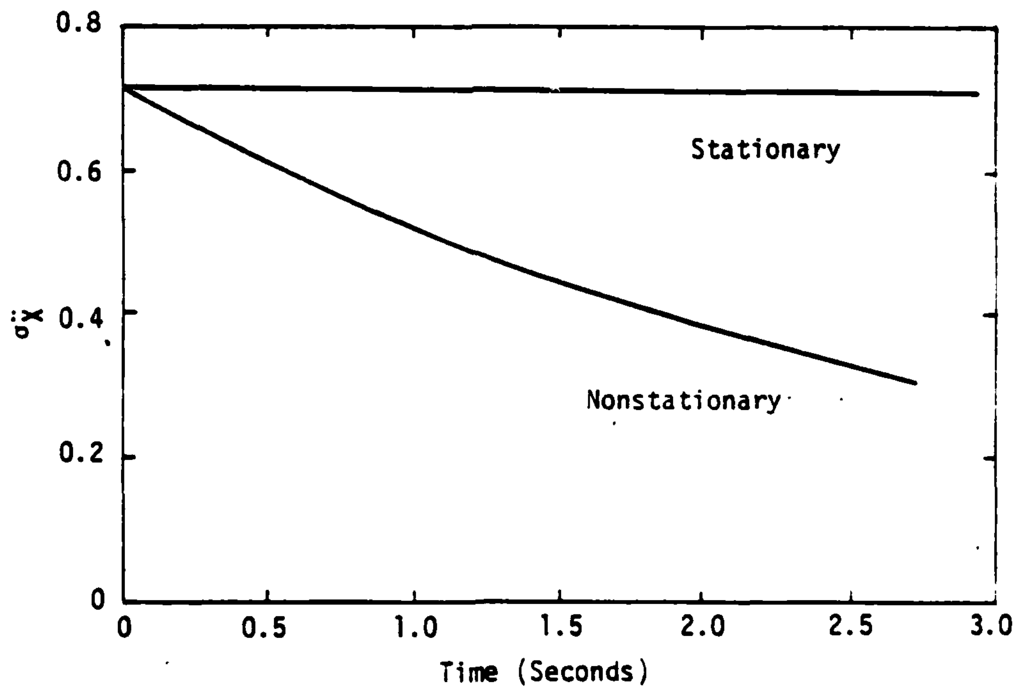

Figure 2. Standard deviation function of input.

TABLE 1. SYSTEM, INPUT, AND COMPUTATION PARAMETERS FOR FIRST PASSAGE PROBLEM

$$\zeta = 0.08$$

$$\Delta t = \pi/(5\omega_n)$$

$$c_a = c_b = \sigma_y = \frac{\Delta t}{4\zeta\omega_n^3}\,\sigma_{\ddot{X}} \quad \text{(= rms value of system response to stationary input)}$$

$$M = 20$$

$$\sigma_{\ddot{X}} = 0.714 \quad \text{(standard deviation of the acceleration input)}$$

curves marked "stationary" are connected with this example; curves marked "nonstationary" are connected to a later example.) The first passage probability is graphed as a function of time in Figure 1. In this example, the first passage probability, at a particular time, is the chance that the displacement response of the structure has passed outside the spatial interval $(-c_a, c_b)$, at or before that time. As expected, the first passage probability increases with time. Other examples, not summarized here, show that when the barrier values are increased in absolute value, the first passage probability function increases more slowly with time. When the barrier values are decreased, the first passage probability increases more rapidly with time.

When the excitation stochastic process is stationary, the first passage probability always approaches unity asymptotically, as time increases. The reason is that a certain portion of the probability for response paths within the barriers is lost to paths outside the barriers at each step in the computation. When the input is nonstationary and decaying, this is not necessarily so.

The second problem to be summarized also concerns the first passage of a base-excited. linear SDF system. In this case, however, the input is a non-stationary random process. The input is a sequence of independent, mean zero, normally distributed random variables with standard deviations that vary as a function of time. Specifically

$$\sigma_{\ddot{X}}(t) = \begin{cases} 0.714\ e^{-0.05\ \omega_n\ t} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{14}$$

This function is graphed in Figure 2 along with the constant value of standard deviation from the previous problem. The first passage bounds were chosen equal in magnitude to those used in the previous problem.

The first passage probability is graphed as a function of time in Figure 1. Since the input used in this problem is less severe than that used in the previous problem, the first passage probability is lower at all times. The first passage probability computed here does not approach the value unity as t increases. This implies that first passage in this system response is not certain. The reason for this behavior is that the input becomes negligible as time progresses.

c. Computational enhancements--The conditional probabilities that displacements y_{k_2}, $k_2 = 1,\ldots,\mu$ will be realized at time t_{j+1}, given the displacement combinations at two previous time steps, y_{k_0} and y_{k_1}, $k_0,k_1 = 1,\ldots,M$ at times t_{j-1} and t_j can be assembled in a computer code as a three-dimensional array. When displacements are discretized into M spatial increments then the array becomes an M by M by M array requiring M^3 storage locations. As discussed previously, this array is treated as a tensor and is multiplied by the state probability matrix which is composed of the probabilities that the displacements actually were at specific discrete locations at the two previous time steps. The multiplication yields the joint probabilities that the displacements are at three locations at times t_{j-1}, t_j, and t_{j+1}, and these probabilities are also stored in an array of size M^3. This multiplication is conceptually shown in Figure 3. The next step in the analytical process is to sum the vectors in the j-1 direction into scalars. This produces an M by M array that is the joint probability that the displacements are at discrete locations at times t_j and t_{j+1}. Note that these form the state probability matrix for the next incremented time step (where $j \rightarrow j-1$ and $j+1 \rightarrow j$). The summing process is conceptually shown in Figure 4.

By simply noting that matrix multiplication is a series of vector operations it can be shown that only a single vector of the conditional probability tensor need be formed at one time. The shaded portions of the conceptual figure illustrate that a single vector of the conditional probability tensor multiplied by the state probability matrix yields a single vector of the joint probability tensor. This vector can be summed into a scalar element of the state probability matrix for the next time step (Figure 4). When this observation is used in the computer coding algorithm, the storage requirements for the computational process are substantially reduced. In particular, the storage requirement is reduced from $2M^3 + M^2$ locations to $M^2 + 2M$ locations.

When the input excitation is stationary, the conditional probability tensor is composed of elements that are constant throughout the computation. Therefore, the approach described above requires the computation of these constants each calculational time step. This means that storage requirements are reduced at an increase in computational time. However, the computations performed in this project were done on a CRAY computer and the elements are formed as a vector operation on the CRAY computer. Therefore, the increased

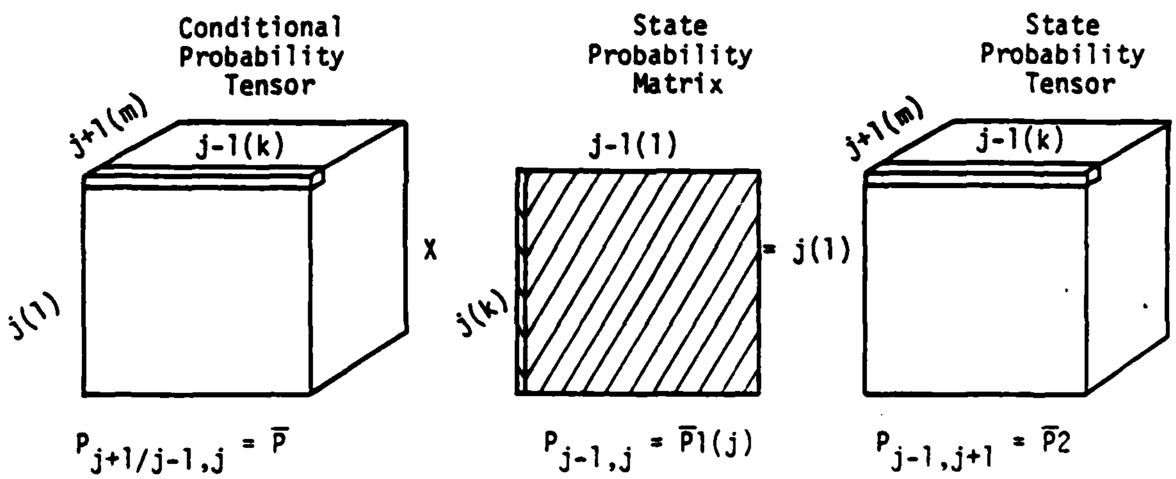

Figure 3. Graphical concept of tensor multiplication in conditional probability analysis.

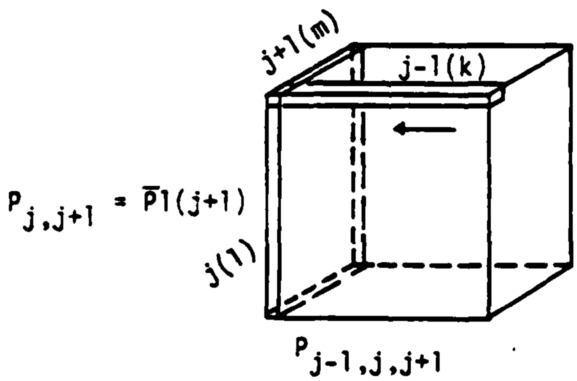

Figure 4. Graphical concept of tensor summation in conditional probability analysis.

computational time is not severe. Furthermore, many of the inputs which are of interest are nonstationary and the conditional probability matrix must be recomputed each time step regardless of the scheme. Therefore, the above change in the calculational process represents an important step in reducing storage requirements. In fact, this general approach may allow the process to be expanded to multiple degrees of freedom without surpassing computer storage requirements.

Other computational enhancements are available. For example, it would be much more efficient to compute first passage probabilities if single-step memory were required rather than the two-step memory shown. Recall, the two-step memory came about as a result of the central difference scheme used to approximate the differential equation. One method to diminish the memory would be to use an approximation that would require only single-step memory, but the accuracy would suffer. Another approach is to identify special features of the response such that the memory may be reduced. An example is that, for lightly damped SDF systems, the response to white noise input is narrow band. That is, given that the response is at a peak, at a given time, the time at which it will be a minimum will be approximately one-half the natural period of the system later. This half-cycle response information can be used to reduce the computational procedure to single-step memory.

To demonstrate this approach the scheme developed above has been used to compute the probability distributions of response amplitudes after a half cycle of response, given that the start conditions are zero velocity and any of the discrete displacements defined in Equation 6. These probabilities are known as half-cycle transition probabilities, and since the analysis considers only those paths remaining in the spatial interval $(-c_a, c_b)$, these probabilities have special meaning. Specifically, these probabilities define the chance that the SDF system response will be at an amplitude value a_{k_1} at the end of a half-cycle and that the response will not pass outside the interval $(-c_a, c_b)$ during the half cycle, given that the response started with velocity zero and an amplitude a_{k_0} at the beginning of the half-cycle.

The sequence of mathematical operations used to obtain the half-cycle transition probabilities is as follows. Set the starting probabilities, Equations 8 and 9, to

$$p_{Y_{-1},Y_0}\left(y_{k_0}, y_{k_1}\right) = \begin{cases} 1, & y_{k_0} = a_{k_0}, \quad y_{k_1} = a_{k_0} \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

Next, divide a response half-cycle (duration π/ω_n) into an integral number of parts, n'. Apply the process n' times, starting with $p_{Y_{-1}Y_0}(y_{k_0}, y_{k_1})$ above to get $P_{Y_{n'-1}Y_n}(y_{k_0}, y_{k_1})$. Finally, apply Equation 10 to obtain the distribution of response values after a half-cycle, considering only those paths which remained between the barriers.

$$p^*\left(Y_{n'} = y_{k_1}\right) = \sum_{k_0=1}^{M} p_{Y_{n'-1}Y_{n'}}\left(y_{k_0}, y_{k_1}\right), \qquad k_1 = 1,\ldots,M \tag{16}$$

Let A_j, $j = 0,\ldots,N'$ be the SDF system amplitude response stochastic process, and let A_j be the random variable in the process denoting the response amplitude at half-cycle j. Then the half-cycle transition probabilities derived using the scheme described above are

$$p^*_{A_{j+1}|A_j}\left(a_{k_1} | a_{k_0}\right) = p^*\left(Y_n' = a_{k_1}\right), \qquad \begin{array}{l} j = 0,\ldots,N' \\ k_0, k_1 = 1,\ldots,M \end{array} \tag{17}$$

where the right-hand side is obtained in Equation 16. Since the computation scheme has been used to compute the transition probabilities given above, it is convenient to choose the values of a_{k_0} and a_{k_1} as those defined in Equation 6. The star (*) superscript is included on the expressions written above to denote the fact that only the response paths remaining in the spatial interval $(-c_a, c_b)$ are included in the computation. Paths which connect a_{k_0} to a_{k_1} in Equation 17, but pass outside $(-c_a, c_b)$ during the half-cycle are neglected.

The transition probabilities presented in Equation 17 provide the same sort of information as provided by the two-step memory transition probabilities in Equation 7. The obvious difference is that here only one memory step is included. The need for one item of information in the memory has been eliminated, since only the response at half-cycle points is considered. At both the beginning and end of every half-cycle the velocity is zero; therefore, the velocity values need not be considered. When a two-step memory is used, both (average) velocity and displacement at times t_{j-1} and t_j influence the response at t_{j+1}. The disadvantage in considering only the response at half-cycle points is that a half-cycle duration must be chosen for use in the computation to obtain the probabilities in Equation 17. This is chosen as π/ω_n for the SDF system. While this is the average half-cycle duration for a system responding to a random input, the actual half-cycle duration is random. Some error is committed in using one value for the half-cycle duration, but when damping is light ($\zeta < 0.20$) the error is small and is more than justified in reducing the memory requirements.

Figure 5a shows the distribution of response amplitudes after a half-cycle of response when the upper and lower limits are chosen as $c_b = 4\sigma_y$ and $-c_a = -4\sigma_y$, and the starting amplitude is chosen as 3.8 σ_y. Also

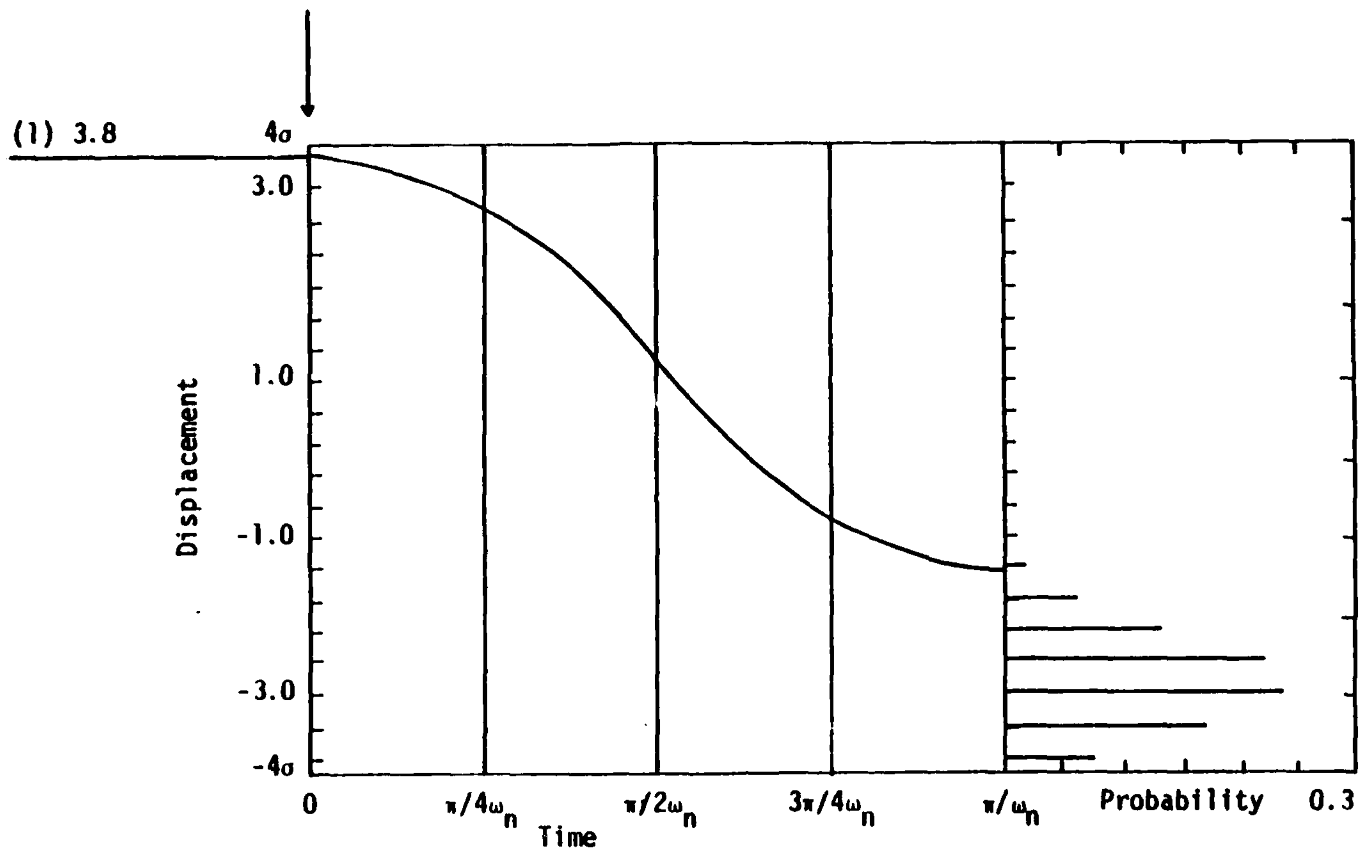

Figure 5a. Half-cycle transition probabilities, $c_a = -c_b = 4\sigma_y$, $a_o = 3.8\ \sigma_y$.

shown is a displacement path the response might execute during the half-cycle. The lines on the right represent the probabilities that the various amplitudes will be realized and the response path will remain in $(-4\sigma_y, 4\sigma_y)$, given that the starting amplitude is $3.8\ \sigma_y$. These probabilities form a vector of transition probabilities given the starting amplitude of $3.8\ \sigma y$.

Figures 5b and 5c show other distributions of amplitude response probabilities given starting amplitudes of $2.2\ \sigma_y$ and $0.2\ \sigma_y$. These graphs present information similar to that described above.

Half-cycle transition probability vectors corresponding to each of the beginning amplitudes can be generated. These vectors can be combined, sequentially, based on the starting amplitude, and the result is a transition probability matrix. Only those paths remaining in the spatial interval $(-c_a, c_b)$ are included. The transition matrix is defined as follows.

$$[P^*_{j+1}] = \begin{bmatrix} p^*_{A_{j+1}|A_j}(a_1|a_1) & p^*_{A_{j+1}|A_j}(a_2|a_1) & \cdots & p^*_{A_{j+1}|A_j}(a_M|a_1) \\ p^*_{A_{j+1}|A_j}(a_1|a_2) & p^*_{A_{j+1}|A_j}(a_2|a_2) & \cdots & p^*_{A_{j+1}|A_j}(a_M|a_2) \\ \vdots & & & \vdots \\ p^*_{A_{j+1}|A_j}(a_1|a_M) & p^*_{A_{j+1}|A_j}(a_2|a_M) & \cdots & p^*_{A_{j+1}|A_j}(a_M|a_M) \end{bmatrix} \qquad (18)$$

The values of the pmf for the various amplitude states at half-cycle zero form a vector $\{P^*_0\}$, where $\{P^*_0\}$ is defined as

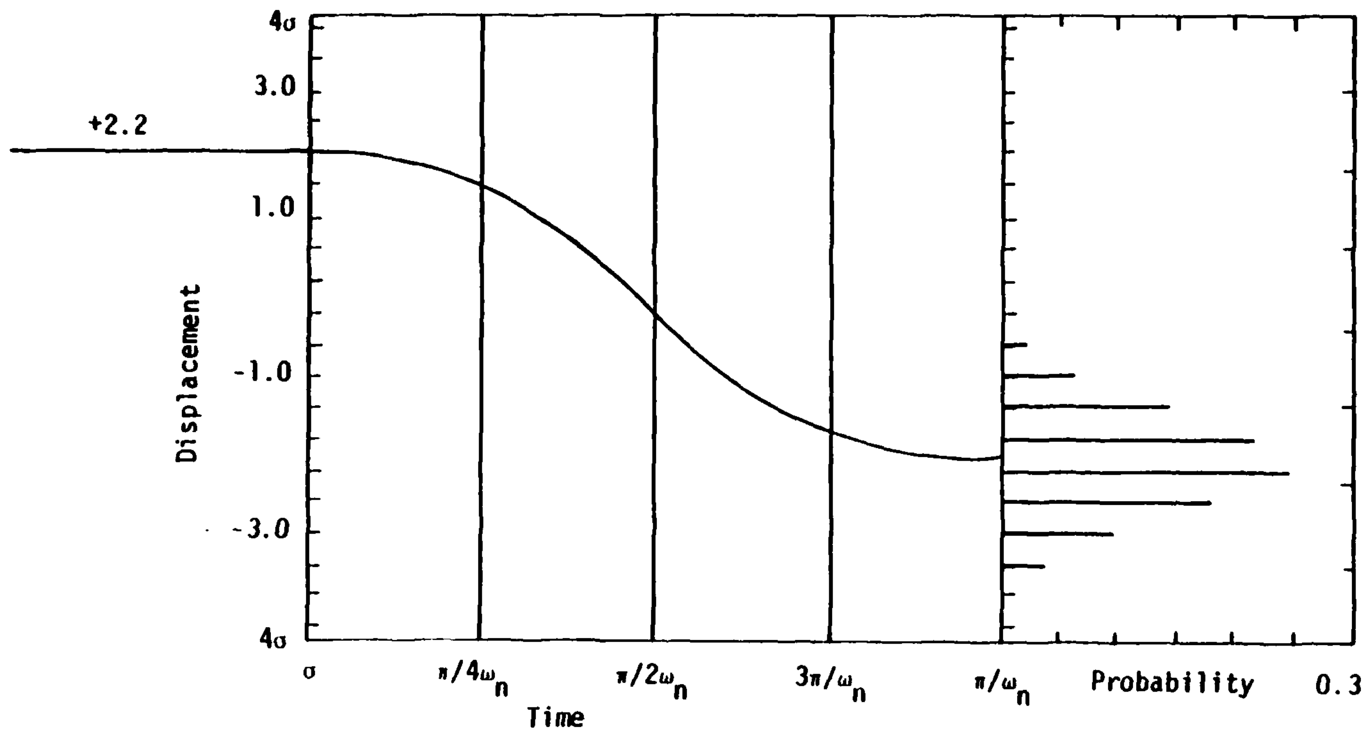

Figure 5b. Half-cycle transition probabilities, $a_0 = 2.2\ \sigma_y$.

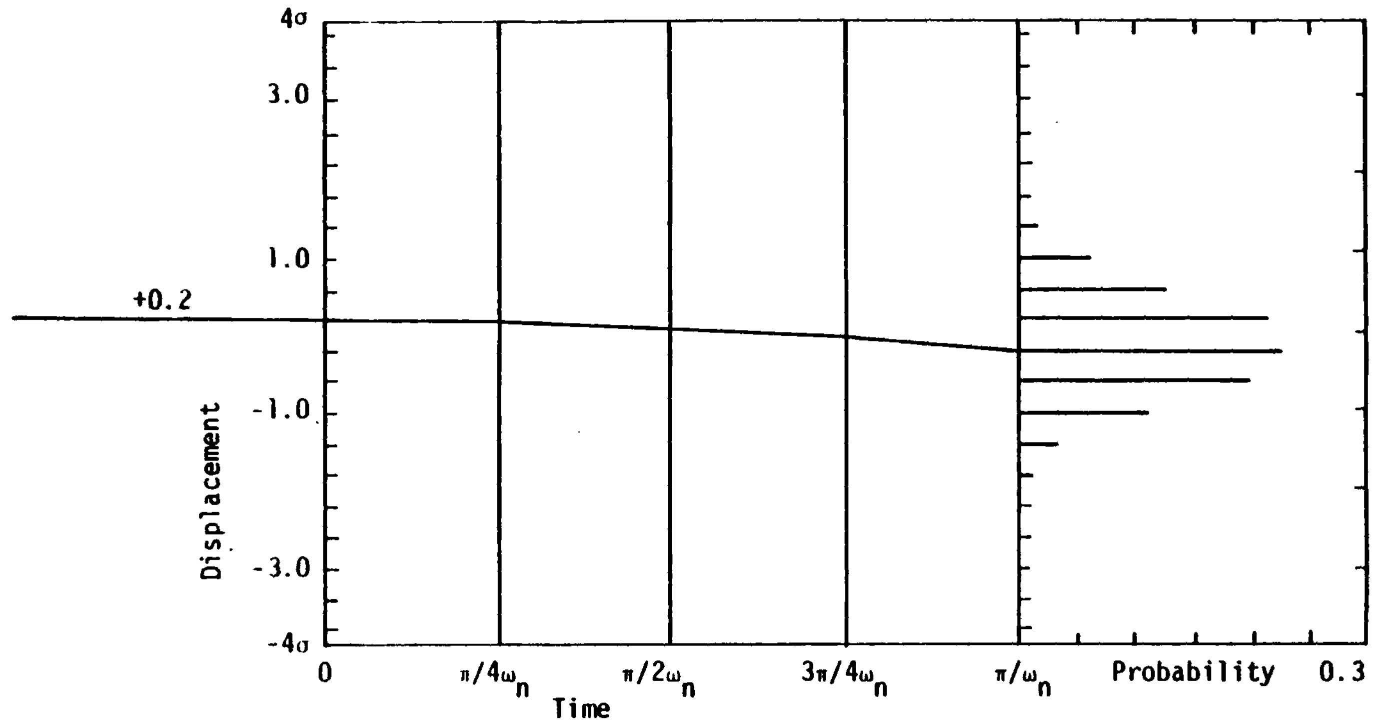

Figure 5c. Half-cycle transition probabilities, $a_0 = 0.2\ \sigma_y$.

$$\{p_0^*\} = \begin{Bmatrix} p_{A_0}(a_1) \\ p_{A_0}(a_2) \\ \vdots \\ p_{A_0}(a_M) \end{Bmatrix} \tag{19}$$

The star (*) superscript indicates that only the probability of those amplitudes within the interval $(-c_a, c_b)$ are included in $\{p_0\}$; therefore, the sum of the elements in $\{p_0^*\}$ may not be one. Then the distribution of amplitudes at half cycle j is

$$\{p_j^*\} = \prod_{k=1}^{j} [P_j^*] \{p_0^*\} , \qquad j = 1,\ldots,N' \tag{20}$$

where only those paths remaining in the spatial interval $(-c_a, c_b)$ are considered in the computation.

The probability that no passage of the SDF system response outside the interval $(-c_a, c_b)$ occurs between half-cycle zero and half-cycle j is

$$P(T_1 > \pi j/\omega_n) = \sum_{k=1}^{M} p_{A_j}(a_k) , \qquad j = 1,\ldots,N' \tag{21}$$

T_1 is the first passage time. $p_{A_j}(a_k)$, $k = 1,\ldots,M$ are the elements in the vector $\{p_j^*\}$. Equation 74 yields the no-passage probability for the time interval $(0, \pi_j/\omega_n)$. In terms of this quantity, the first passage probability can be written

$$P(T_1 \leq \pi j/\omega_n) = 1 - P(T_1 > \pi j/\omega_n) , \qquad j = 1,\ldots,N' \tag{22}$$

The first passage probability can be computed as a function of time when the procedures outlined above are followed. The computations performed using these half-cycle transition probabilities are subject to the assumptions made in the previous section.

When the input excitation stochastic process is stationary in time, the present technique for analysis of first passage probabilities of a linear, base-excited SDF system will yield a computational time savings proportional to MN/N'. This factor will be considerable when the spatial discretization is fine (M is large) and the time discretization is fine (N/N' is large and both analyses cover the same interval of time). When the input is nonstationary, the present technique does not result in computational time savings.

Expansion of the above analytic approach to highly nonlinear or MDF problems is not direct. The basic assumption of narrow band response would not in general be true in these cases. However, the concept of identifying special features of the response or the computational scheme holds. For example, because of stability, the displacement of a node in a finite element scheme is bounded by the time step. There are certain displacements that have absolutely zero probability of being reached in a single time step. Therefore, it may be possible to extend the above concept into solving the much more difficult problems of nonlinear MDF problems.

3. FIRST PASSAGE OF NONLINEAR ELASTIC SDF SYSTEMS

a. *Theoretical analysis*--Once the first passage problem for a linear elastic SDF system has been solved, it is a relatively simple matter to extend the analysis to nonlinear elastic SDF systems. The analysis presented above is extended in this section to include the nonlinear elastic case. The input used to excite the SDF system is the same as that used previously; that is, the input is a sequence of independent, mean zero random variables.

The equation of motion governing the response of a nonlinear elastic, base-excited SDF system can be written

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 \left(\sum_{\ell=1}^{m} c_\ell \, y^{(2\ell-1)} \right) = -\ddot{x} \tag{23}$$

where y is the relative displacement response of the SDF system, $\ddot{x}$ is the base acceleration, ω_n is the "small displacement response" natural frequency of the system, ζ is the damping factor and dots denote differentiation with respect to time. This equation governs the response of a system whose restoring force is an odd polynomial function, of degree $2m - 1$.

To obtain from this equation the transition probabilities required for a probabilistic response analysis, first discretize the equation in time. When the central difference approximations are used to replace the derivatives in the above expression, then

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{\Delta t^2} + 2\zeta\omega_n \frac{y_{j+1} - y_{j-1}}{2\Delta t} + \omega_n^2 \left(\sum_{\ell=1}^{m} c_\ell \, y_j^{(2\ell-1)} \right) = \ddot{x}_j \, , \qquad j = 0, \ldots, N \tag{24}$$

This equation governs system motion at times $t_j = j\Delta t$, $j = 0, \ldots, N$. Rearrangement of this equation yields an expression for y_{j+1}.

$$y_{j+1} = (1 + \zeta\omega_n \Delta t)^{-1} \left[\Delta t^2 \ddot{x}_j + \left(\left(\Delta t^2 \omega_n^2 \sum_{\ell=1}^{m} c_\ell y_j^{(2\ell-1)} \right) - 2 \right) y_j + (1 - \zeta\omega_n \Delta t) \, y_{j-1} \right], \qquad j = 0, \ldots, N \tag{25}$$

When a nonlinear elastic system is excited by an input stochastic process, the response at t_{j+1} is approximately governed by the above equation if the system parameters are constants. To represent the approximate stochastic relationship between input and response, the lower case y_js must be replaced with upper case Y_js and the lower case $\ddot{x}_j$ with $\ddot{X}_j$. Then using the definition of conditional probability, an expression for the SDF system response transition probabilities can be obtained. That is,

$$P\left(y_a < Y_{y+1} \leq y_b \middle| Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\right)$$

$$= P\left(\frac{y_a - 2y_{k_1} + y_{k_0}}{\Delta t^2} + 2\zeta\omega_n \frac{y_a - y_{k_0}}{2\Delta t} + \omega_n^2 \left(\sum_{\ell=1}^{m} c_\ell \, y_{k_1}^{(2\ell-1)}\right)\right.$$

$$< \ddot{X}_j \leq \frac{y_b - 2y_{k_1} + y_{k_0}}{\Delta t^2} + 2\zeta\omega_n \frac{y_b - y_{k_0}}{2\Delta t} + \omega_n^2 \left(\sum_{\ell=1}^{m} c_\ell \, y_{k_1}^{(2\ell-1)}\right)$$

$$\left.\middle| Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\right), \qquad j = 0,\ldots,N \tag{26}$$

This expression is the nonlinear system equivalent to the transition probability expression derived in Section II.2.a for linear systems; and it is derived in the same way and can be used in exactly the same way. It is noted that not only stability of the finite difference equation must be properly addressed, but that adequate time step is selected to propagate probability outside initial conditions.

When the system response displacements are discretized, Equation 26 can be used to obtain the transition probabilities for the discrete-time/discrete-space SDF system response stochastic process. The assumptions and notations regarding response displacement bounds are the same here as those used in Section II.2.a. The discrete displacements in $(-c_a, c_b)$ are y_k, $k = 1, \ldots, M$ defined in Equation 6. The system response transition probabilities are

$$P\left(Y_{j+1} = y_{k_2} \middle| Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\right)$$

$$= P\left(y_{k_2} - \Delta y/2 < Y_{j+1} \leq y_{k_2} + \Delta y/2 \middle| Y_{j-1} = y_{k_0}, Y_j = y_{k_1}\right), \quad \begin{array}{l} j = 0,\ldots,N \\ k_0,k_1,k_2 = 1,\ldots,M \end{array} \tag{27}$$

where the expression on the right-hand side is evaluated using Equation 26.

Now when the starting joint pmf of the response stochastic process is specified as in Equation 8 or 9 and the probability computation is marched out in time, as before, first passage results equivalent to those obtained in Section II.2.a are obtained for the present problem. The analysis required here is precisely equivalent to that described in Section II.2.a, following Equation 9. Some numerical examples are presented in the following section to demonstrate the results of this analysis.

In order to use the present approach for the practical failure analysis of a real system, it is necessary that (1) the system be accurately modeled as a nonlinear elastic, SDF system, and (2) failure occurs due to peak response.

The computational enhancements discussed in Section II.2.c can be applied in the nonlinear first passage analysis. The first technique discussed in that section can be applied without modification. The second computation scheme, the conversion of the response random process from a conditional probability random process with two-step memory to one with one-step memory, can be applied here, but only in a restricted way. An underlying assumption in the establishment of that computation scheme is that the response will be narrow band. When the degree of nonlinearity in SDF system response is not too great, and when system damping is light, then the response will be narrow band. When the response nonlinearity becomes great, the response will cease to be narrow band. Ductility ratios of greater than

1.5 and damping coefficients of greater than 20 percent of critical would be considered limits for these approaches.

The following analyses characterizing the response of nonlinear SDF systems will make the assumption of narrow band response and use the computational procedures previously discussed.

b. Numerical examples--The first computer program listed in the appendix, and used to solve the linear first passage problems, can also be used to solve nonlinear elastic first passage problems. Some numerical examples demonstrating the capability of the computer program to solve nonlinear first passage problems are presented in this section.

The computer program computes the probability of first passage of the response of a nonlinear elastic, base-excited, SDF system, outside the spatial interval $(-c_a, c_b)$. The response is excited by a stochastic process input composed of a sequence of zero mean, normal random variables. Zero start initial conditions for the SDF system are assumed.

Three first passage problems are presented in this section. The system parameters used in Equation 23 through 27 are summarized in Table 2. The input is a band-limited white noise whose standard deviation is given in Table 2. The first passage probability was computed for each of the three systems (linear, hardening, and softening), and the results are shown in Figure 6.

TABLE 2. SYSTEM, INPUT AND COMPUTATION PARAMETERS FOR NONLINEAR FIRST PASSAGE PROBLEMS

$\zeta = 0.08$	Linear system: $m = 1$, $c_1 = 1$
$\Delta t = \pi/(5\omega_n)$	Nonlinear hardening system: $m = 2$, $c_1 = 1$, $c_2 = 1.5$
$M = 20$	Nonlinear softening system: $m = 2$, $c_1 = 1$, $c_2 = 1.5$
$\sigma_{\ddot{X}} = 0.714$	
$c_a = c_b = \sigma_y = \dfrac{\Delta t}{4\zeta\omega_n^3}\,\sigma_{\ddot{X}}$	

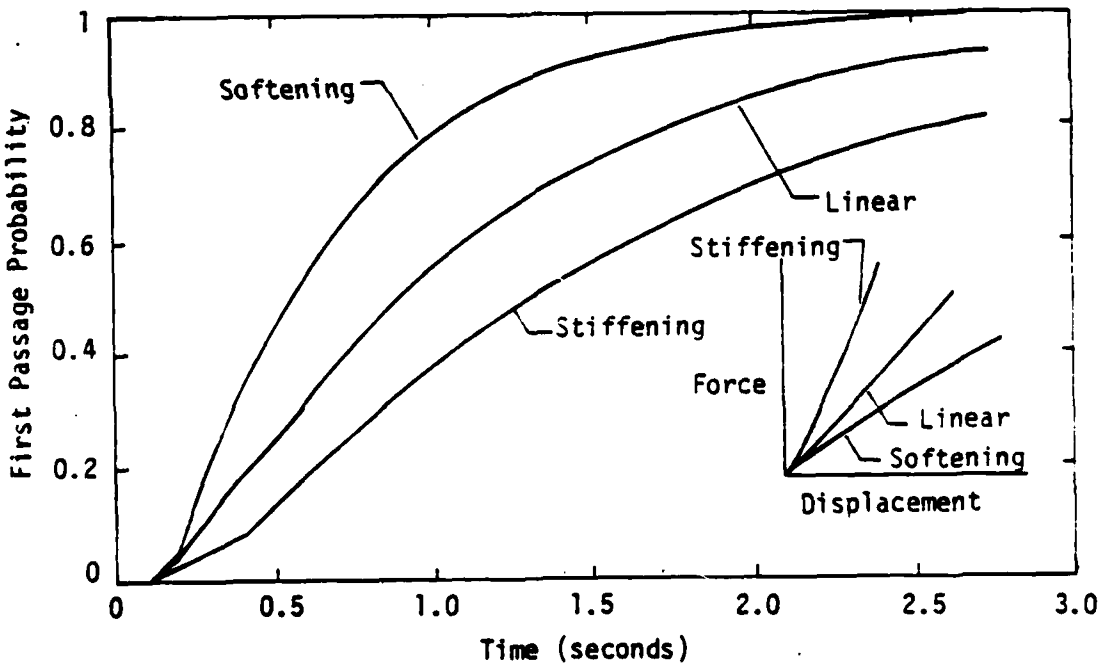

Figure 6. First passage probability for linear, hardening and softening SDF systems $\omega_n = 2\pi$.

The hardening system tends to have lower response amplitudes than the linear system; therefore, the first passage probability curve of the hardening system is below the first passage probability curve of the linear system. The softening system tends to exhibit higher response amplitudes than the linear system; therefore, the first passage probability curve of the softening system falls above the first passage probability curve of the linear system.

4. *PROBABILITY DISTRIBUTION OF SYSTEM RESPONSE*

a. Theoretical analysis--Up to this point only the first passage probability has been used to characterize structural response. The first passage problem has been pursued since (1) peak response can be an important criterion in predicting system failure, and (2) relatively little work has been done toward solving first passage problems. On the other hand, much work has been done toward characterizing the probability distribution of the overall response of a linear system. This section modifies the general approach outlined in Sections II.2 and II.3 to compute the approximate probability distribution of the overall response, rather than just the first passage probability.

The main reason it is difficult to compute the probability distribution of the overall structural response is that the procedures outlined in the previous sections are ideally suited to tracking a portion of the response probability remaining between finite barriers. The probability connected with paths passing outside the barriers is discarded in succeeding computations. When the probability distribution of the overall response is of interest, all paths must be accounted for and none of the probability can be discarded, no matter how great the amplitudes of the paths involved.

Since the computational scheme used with the present approach cannot account for all paths, especially those whose amplitudes are very large, a compromise is made by taking two actions: (1) by setting the barriers of the spatial interval $(-c_a, c_b)$ at very large amplitudes so that the probabilities associated with paths crossing outside the barriers are low, and (2) by normalizing the probabilities to a value of 1 at every step of the computation; that is, by changing the probabilities, $p^*_{Y_{j-1}Y_j}(y_{k_0}, y_{k_1})$, referred to in Equation 10. On the jth computation interval, the sum

$$S^* = \sum_{k_0=1}^{M} \sum_{k_1=1}^{M} p^*_{Y_{j-1}Y_j}\left(y_{k_0}, y_{k_1}\right) \tag{28}$$

is the no-passage probability through computation cycle j. If $p^*_{Y_{j-1}Y_j}(y_{k_0}, y_{k_1})$ is modified by multiplying it by $(S^*)^{-1}$, the result is

$$p_{Y_{j-1},Y_j}\left(y_{k_0}, y_{k_1}\right) = (S^*)^{-1}\, p_{Y_{j-1},Y_j}\left(y_{k_0}, y_{k_1}\right), \qquad k_0,k_1 = 1,\ldots,M \tag{29}$$

The path probabilities have been modified so that the chance that the response remains in $(-c_a, c_b)$ is one. When the interval $(-c_a, c_b)$ is wide enough, the probability distribution of the response will be accurate. The star (*) has been dropped on the left, above, since now the probability for all response paths has been accounted for, though in an approximate way.

The distribution of the response can now be computed at each time point, and is obtained by summing the expression in Equation 29 over all k_0.

$$p_{Y_j}\left(y_{k_1}\right) = \sum_{k_0=1}^{M} p_{Y_{j-1},Y_j}\left(y_{k_0}, y_{k_1}\right), \qquad k_1 = 1,\ldots,M \tag{30}$$

Of course, the sum of $p_{Y_j}(y_{k_1})$ over all k_1 equals 1.

The value of the above modification in computation scheme does not lie in the capability it establishes to approximate the distribution of the response of a linear SDF system. This capability already exists using a theoretical probability analysis of linear continuous systems. Rather, its value lies in the fact that the present analysis can be extended to nonlinear elastic and hysteretic systems. This extension will be outlined in following sections. The fact that the method presented above yields results corresponding to an exact analysis will be used to check the accuracy of the present approach.

b. Numerical example--The analysis outlined in Section II.4.a is checked for accuracy in this numerical example. Three numerical examples were solved. In each example the response probability distribution was computed. The input is a band-limited white noise random process whose standard deviation is given in Table 3. After the response reached a steady state, the response displacement pmf was recorded. The parameters of the three problems solved are given in Table 3. In each problem the lower and upper spatial limits are defined in terms of rms response, once it has reached the stationary state.

Figure 7 shows plots of the response cdf's for each problem. Also shown is the theoretical cdf for the linear SDF structural response. The statistics of the theoretical response are available, for example, in Reference 1. The plots show that there is reasonable agreement between the theoretical results and the results obtained in Section II.4.a.

5. DISTRIBUTION OF DISSIPATED ENERGY IN AN ELASTO-PLASTIC SYSTEM

a. Theoretical analysis--The analysis of structural reliability hinges on our ability to identify the modes of failure which might lead to the

TABLE 3. SYSTEM, INPUT AND COMPUTATION PARAMETERS FOR RESPONSE PROBABILITY DISTRIBUTION PROBLEMS

$\Delta t = \pi/5\omega_n$
$\sigma_{\ddot{X}} = 0.714$
$\zeta = 0.08$
Problem 1
$M = 10$
$c_b = -c_a = 2\sigma_y = 2 \dfrac{\Delta t}{4\zeta\omega_n^3} \sigma_{\ddot{X}}$
Problem 2
$M = 20$
$c_b = -c_a = 4\sigma_y$
Problem 3
$M = 40$
$c_b = -c_a = 8\sigma_y$

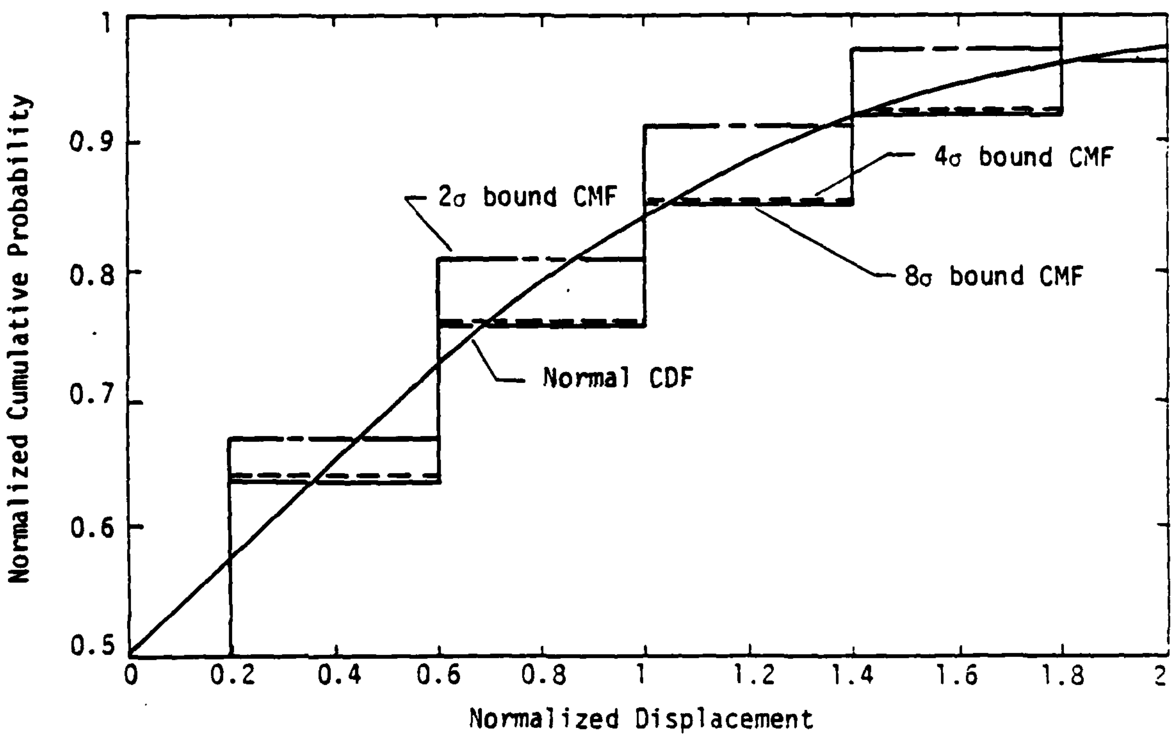

Figure 7. CDF of linear system responses.

demise of a structural system. Modes of structural failure can be qualitatively identified, and, to some extent, quantitative predictions of failure can be made. But to date, the time of failure cannot be exactly predicted, even when the response history is computed exactly. The reason is that material behavior is random. Yet in a reliability analysis some criterion must be chosen to judge whether or not failure occurs in a response. The characterization of the criterion can be deterministic or probabilistic.

Consider the elasto-plastic SDF system. Various criteria can be used to judge failure in this system. For example, accumulated plastic deformation can be used to determine when failure will occur, or a permanent set can be used as a failure criterion. The amount of energy dissipated by the system can be used as a failure criterion, or a combination of these factors can be used as a failure criterion. In the present analysis, the amount of energy dissipated by the system is considered. There are two reasons for this: (1) the analysis of probability distribution and first passage probabilities for this quantity is slightly more difficult than the equivalent analysis for the other quantities; and (2) this might serve as a damage criterion for an actual system.

The equation governing motion of a base-excited, elasto-plastic, SDF system is

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 R(y) = -\ddot{x} \tag{31}$$

where y is the relative displacement of the system mass, x is the base acceleration, ω_n is the small displacement natural frequency of the system, ζ is the system viscous damping factor, kR(y) is the elasto-plastic spring restoring force function, k is the system small displacement stiffness, and dots denote differentiation with respect to time. An example of a realization of

$R(y)$ is shown in Figure 8. When the response displacement is small, $R(y)$ is linear; but when y exceeds the yield displacement D, the restoring force becomes constant, kD. This continues until the sign of the velocity changes. Then response starts to take place along a new straight line in the force-displacement space. This continues until yielding occurs again, etc.

As stated earlier, the response of a linear SDF system to broad-band random excitation is a narrow-band random process. This means that when the input has power over a range of frequencies wider than the bandwidth of the SDF system and including its natural frequency, the system response displays power content mainly in the band of frequencies nearly surrounding ω_n. The realizations of the response stochastic process resemble a sinusoidal signal with randomly varying amplitude. That is, the response displays regularly occurring peaks and troughs spaced at a time interval of about π/ω_n. This behavior also occurs in elasto-plastic systems when yielding is not too great. In fact, a half-cycle of response might be described by the curve shown in Figure 9.

The starting amplitude of the response is a_{j-1} and the final amplitude is a_j. Assume that a half-cycle in the response of an elasto-plastic system is characterized by the curve shown in Figure 9 and that this curve can be approximated by

$$y(t) = \frac{1}{2}(a_{j-1} + a_j) - \frac{1}{2}(a_j - a_{j-1}) \cos \omega_n t \ , \qquad 0 \le t \le \pi/\omega_n \tag{32}$$

If the dashed line in Figure 9 represents the level at which yielding occurs in the response, then a portion of the response during the half-cycle occurs in the yield range.

The response can also be depicted in the elasto-plastic spring restoring force versus displacement space. When it is, the system executes the curve shown in Figure 10.

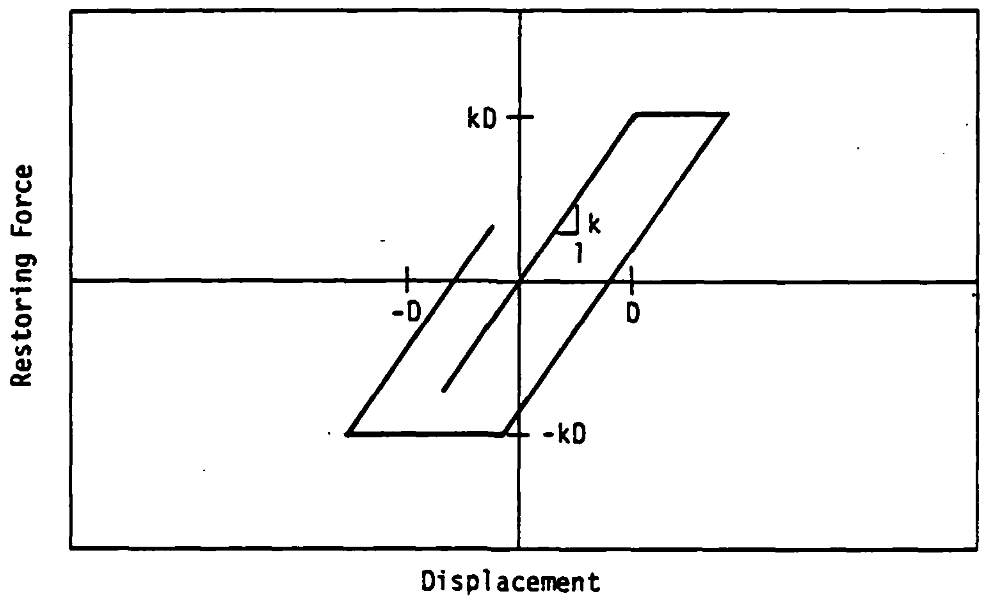

Figure 8. Force-displacement curve for an elasto-plastic system.

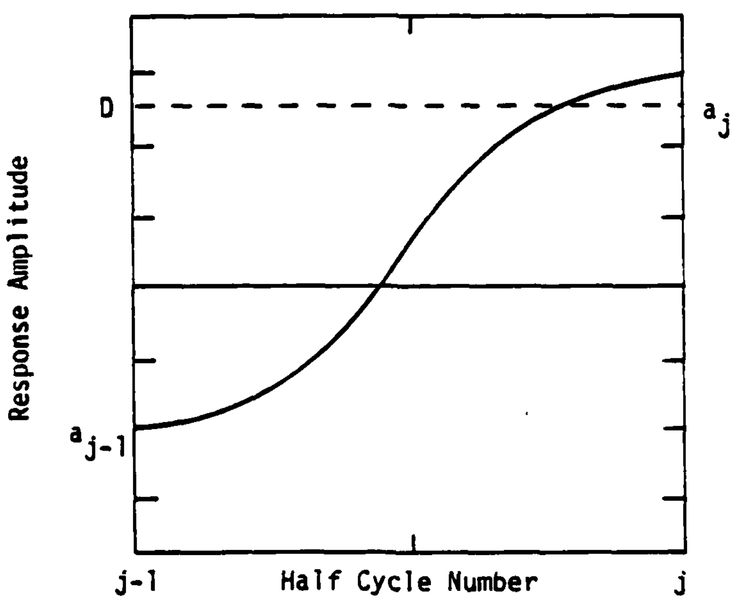

Figure 9. Half-cycle of narrow-band response in the space of displacement versus time.

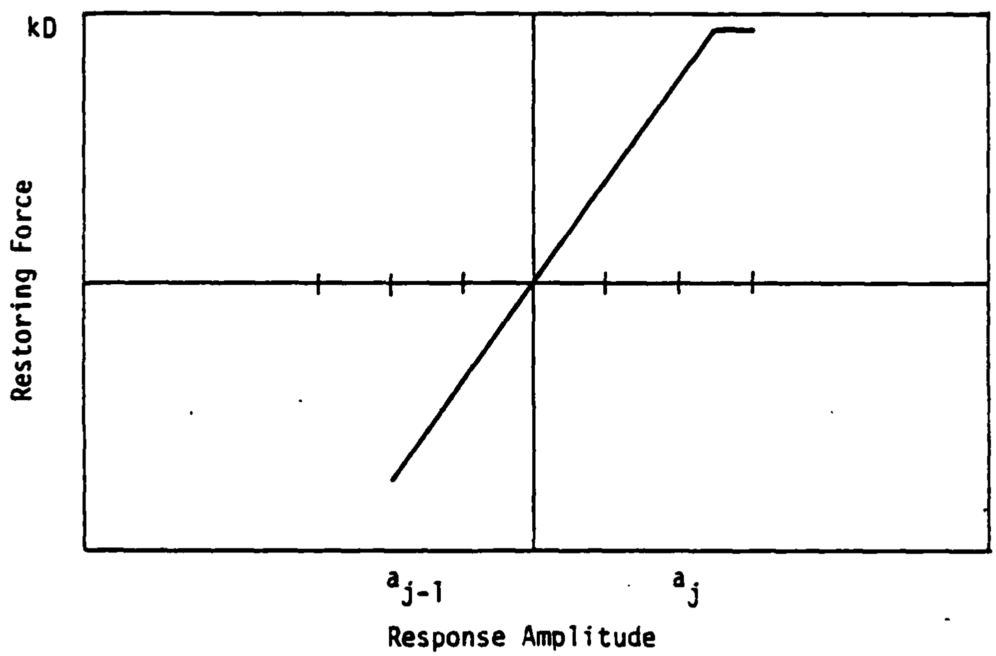

Figure 10. Half-cycle of elasto-plastic system response in the space of restoring force versus displacement.

Energy is dissipated in two ways during the response half-cycle. It is dissipated in the elasto-plastic spring and in the damper. The energy dissipated in the spring during a half-cycle equals the net area under the force displacement curve. This area depends on the starting and ending amplitudes, and the SDF system parameters. The energy dissipated in the spring can be expressed

$$E_{Ds} = \begin{cases} \frac{k}{2}\,(a_j^2 - a_{j-1}^2) & a_j \leq D \\ \frac{k}{2}\,(D^2 - a_{j-1}^2) + kD(a_j - D) & a_j > D \end{cases} \tag{33}$$

This quantity clearly depends on the starting and ending amplitudes and on the system parameters. The dissipated energy can be negative, but only when the amplitude decreases.

The energy dissipated by the damper during a half-cycle of response can be expressed

$$E_{Dd} = c\int_{y(0)}^{y(\pi/\omega_n)} \dot{y}\; dy \tag{34}$$

The response velocity can be obtained by differentiating $y(t)$ in Equation 32, and the variable of integration can be changed in Equation 163 to obtain

$$E_{Dd} = c\int_{0}^{\pi/\omega_n} (\dot{y})^2\; dt$$

$$= c\omega_n\pi(a_j - a_{j-1})^2/8 \tag{35}$$

The energy dissipated in the damper is always positive, and depends on the starting and ending amplitude of the half-cycle response and the system parameters.

In the analysis of Section II.2.c the half-cycle transition probabilities were obtained for a linear system. In Section II.3.a, a method for normalizing the transition probabilities to approximately account for all the response paths was developed. Using the methods established in those sections, half-cycle transition probabilities can be obtained for a nonlinear elastic system, based on the analysis of Section II.4.a. These are

$$p_{A_j|A_{j-1}}\left(a_{k_1}\middle|a_{k_0}\right) = P\left(A_j = a_{k_1}\middle|A_{j-1} = a_{k_0}\right), \qquad j = 1,\ldots,N' \tag{36}$$

Because of the normalization, the sum of $p_{A_j|A_{j-1}}(a_{k_1}|a_{k_0})$ over all k_1 is 1.

According to the discussion given above, when a nonlinear system whose force-displacement diagram resembles that given in Figure 10 (and whose damping value is c) executes a half-cycle of response, it dissipates the energy given by the sum of Equations 33 and 35. The energy dissipated is a function of a_{j-1} and a_j.

Let ΔE_j be the random variable denoting the energy dissipated during a half-cycle of elasto-plastic, SDF system response. Then

$$\Delta E_j = E_{Ds} + E_{Dd} \tag{37}$$

Since the energy dissipated in the spring and damper depends on the amplitudes a_{j-1} and a_j, and since the amplitudes of the response are random, ΔE_j is also random. The probability that the realized value of ΔE_j during a half-cycle of response falls within a given interval is dependent upon the joint distribution of A_{j-1} and A_j, the random amplitudes at either end of the half-cycle. To compute the probability distribution of energy dissipated as a function of time, proceed in the following manner. First, compute the probability distribution of energy dissipated during each half-cycle of response. Then use this information to compute the probability distribution of energy dissipated from the start of the response through the jth half-cycle of response.

Define a stochastic process E_j, $j = 0,\ldots,N'$ as the cumulative energy dissipated stochastic process. It measures the energy dissipated by a base excited, elasto-plastic, SDF system at half cycles indexed zero through N'. The cumulative amount of energy dissipated through half cycle j depends on the amount of energy dissipated at half cycle j-1 and the increment of energy dissipated during the half-cycle. This latter quantity is governed by the distribution of ΔE_j, $j = 1,\ldots,N'$.

To compute the distribution of ΔE_j, assume that during each half-cycle of response the amplitude changes sign. This follows from the assumption of narrow bandedness of the response. This means that in the analysis only the possibility that the response will proceed from a trough to a peak, or from a peak to a trough, is permitted. In Figure 9, a_{j-1} and a_j will always have opposite signs. The discrete amplitudes defining possible values of the response are denoted a_k, $k = 1,\ldots,M$. All the a_k are positive numbers, and when the system goes through a trough it simply assumes an amplitude whose value is the negative of one of the a_k values. The first M_1 of the values a_k, $k = 1,\ldots,M$ where $M_1 \leq M$, denote the response values the system can assume where yielding does not occur. (Recall that, for accuracy, a_M must be chosen large enough so that the possibility that the response assumes that value is small.) When a_{max} is the largest amplitude to be represented in the probabilistic response analysis, the discrete amplitudes can be defined as

$$a_k = (k - \frac{1}{2})\,\Delta a\ , \qquad k = 1,\ldots,M \tag{37}$$

where

$$\Delta a = a_{max}/M \tag{37a}$$

Using Equations 33 and 35, compute the maximum and minimum values the dissipated energy can assume during a half-cycle of response. These are denoted e_{max} and e_{min} and are given by

$$e_{max} = \frac{k}{2}\left(D^2 - \left(-a_{M_1}\right)^2\right) + kD\left(a_M - D\right) + c\omega_n \pi\left(a_M - \left(-a_{M_1}\right)\right)^2/8 \tag{38a}$$

$$e_{min} = \frac{k}{2}\left(a_1^2 - (-a_1)^2\right) + c\omega_n \pi(a_1 - (-a_1))^2/8 \tag{38b}$$

where the response at half-cycle j-1 is required to begin in the elastic range and can end, at half-cycle j, as high as a_M.

The range of values the dissipated energy may assume can now be discretized. The discretization increment is defined

$$\Delta e = \frac{e_{max} - e_{min}}{M'} \tag{39a}$$

The discrete values that dissipated energy may assume can be defined

$$e_k = (k - \tfrac{1}{2})\,\Delta e + e_{min}\,, \qquad k = 1,\ldots,M' \tag{39b}$$

The e_k are uniformly distributed in the interval (e_{min}, e_{max}).

During any actual half-cycle of response, the energy dissipated assumes one value in a continuous range of values. Using the present computation scheme the range of values is (e_{min}, e_{max}). Assume that when the energy dissipated during a half-cycle of response falls in the interval $(e_k - \Delta e/2,\ e_k + \Delta e/2)$, it can be accurately represented by e_k. The accuracy of this assumption is good when M' is large enough.

Now recall that the amplitude transition probabilities (Equation 35) were obtained using the same type of discretization described above. The discrete amplitudes are a_k, $k = 1,\ldots,M$ and the amplitude increment is Δa. When the response falls in the interval $(a_k - \Delta a/2,\ a_k + \Delta a/2)$, assume that the response is a_k. For each pair of amplitudes $-a_{k_0}$, $k_0 = 1,\ldots,M_1$ at half-cycle j-1, and a_{k_1}, $k_1 = 1,\ldots,M$ at half-cycle j, compute the range of possible energies dissipated during the half-cycle; the starting amplitude is a_{k_0}; the ending amplitude is in the range $(a_{k_1} - \Delta a/2,\ a_{k_1} + \Delta a/2)$. The range of energy dissipated values is computed using Equations 86 and 88 and can be denoted $(e^{min}_{k_0k_1},\ e^{max}_{k_0k_1})$ where

$$e^{min}_{k_0k_1} = \begin{cases} \frac{k}{2}\left(\left(a_{k_1} - \Delta a/2\right)^2 - a^2_{k_0}\right) + c\omega_n \pi\left(a_{k_1} - \Delta a/2 - a_{k_0}\right)^2/8\ , & a_{k_1} \le D. \\ \frac{k}{2}\left(D^2 - a^2_{k_0}\right) + kD\left(a_{k_1} - \Delta a/2 - D\right) + c\omega_n \pi\left(a_{k_1} - \Delta a/2 - a_{k_0}\right)^2/8\ , & a_{k_1} > D \end{cases} \quad (40a)$$

$$e^{max}_{k_0k_1} = \begin{cases} \frac{k}{2}\left(\left(a_{k_1} + \Delta a/2\right)^2 - a^2_{k_0}\right) + c\omega_n \pi\left(a_{k_1} + \Delta a/2 - a_{k_0}\right)^2/8\ , & a_{k_1} \le D \\ \frac{k}{2}\left(D^2 - a^2_{k_0}\right) + kD\left(a_{k_1} + \Delta a/2 - D\right) + c\omega_n \pi\left(a_{k_1} + \Delta a/2 - a_{k_0}\right)^2/8\ , & a_{k_1} > D \end{cases} \quad (40b)$$

The range of values $(e^{min}_{k_0k_1},\ e^{max}_{k_0k_1})$ will not, in general, equal any one of the ranges $(e_k - \Delta e/2,\ e_k + \Delta e/2)$, $k = 1,\ldots M'$. The probability $p_{A_j|A_{j-1}}(a_{k_0}|a_{k_1})$, of Equation 89, can be associated with the chance that the response will start at an amplitude $-a_{k_0}$ (taken from one of the values $-a_k$, $k = 1,\ldots,M_1$) and end at an amplitude a_{k_1}, with energy dissipated falling in the interval $(e^{min}_{k_0k_1},\ e^{max}_{k_0k_1})$.

Based on the above arguments, the joint, conditional pmf of amplitude and energy dissipation increment can be developed. Let

$$p_{\Delta E_j A_j | A_{j-1}}\left(e_{k_2}, a_{k_1} \middle| -a_{k_0}\right), \quad \begin{array}{l} k_0 = 1,\ldots,M_1 \\ k_1 = 1,\ldots,M \\ k_2 = 1,\ldots M' \end{array}$$

denote the conditional joint pmf of ΔE_j and A_j, given A_{j-1}. That is

$$p_{\Delta E_j A_j | A_{j-1}}\left(e_{k_2}, a_{k_1} \middle| -a_{k_0}\right) = P\left(\Delta E_j = e_{k_2}, A_j = a_{k_1} \middle| A_{j-1} = -a_{k_0}\right),$$

$$\begin{array}{l} k_0 = 1,\ldots,M_1 \\ k_1 = 1,\ldots,M \\ k_2 = 1,\ldots,M' \\ j = 1,\ldots,N \end{array} \qquad (41)$$

The values of the pmf are obtained in the framework of a digital program computation scheme as follows. (1) All the values of the pmf, $p_{\Delta E_j A_j | A_{j-1}}(e_{k_2}, a_{k_1} | -a_{k_0})$, are set to zero. (2) Set $k_0 = 1$ and $k_1 = 1$. (3) Compute $(e^{min}_{k_0 k_1}, e^{max}_{k_0 k_1}) = (e^{min}_{11}, e^{max}_{11})$ and $p_{A_j | A_{j-1}}(a_{k_1} | -a_{k_0}) = p_{A_j | A_{j-1}}(a_1 | -a_1)$ using Equations 40 and 36. (4a) When the interval $(e^{min}_{11}, e^{max}_{11})$ is completely included in one of the intervals $(e_\ell - \Delta e/2, e_\ell + \Delta e/2)$, then increment the pmf $p_{\Delta E_j A_j | A_{j-1}}(e_\ell, a_1 | -a_1)$ by the amount $p_{A_j | A_{j-1}}(a_1 | a_1)$. (4b) When the interval $(e^{min}_{11}, e^{max}_{11})$ overlaps two or more energy increment intervals, say $(e_{\ell_1} - \Delta e/2, e_{\ell_1} + \Delta e/2)$ through $(e_{\ell_2} - \Delta e/2, e_{\ell_2} + \Delta e/2)$, then the probability, $p_{A_j | A_{j-1}}(a_1 | -a_1)$ is divided into fractional parts where the fractions are linearly proportional to the fractions of the intervals, $(e_{\ell_1} - \Delta e/2, e_{\ell_1} + \Delta e/2)$ through $(e_{\ell_2} - \Delta e/2, e_{\ell_2} + \Delta e/2)$, spanned by the interval $(e^{min}_{11}, e^{max}_{11})$. These fractions of $p_{A_j | A_{j-1}}(a_{k_1} | -a_{k_0})$ are used to increment the values $p_{\Delta E_j A_j | A_{j-1}}(e_{\ell_1}, a_{k_1} | -a_{k_0})$ through $p_{\Delta E_j A_j | A_{j-1}}(e_{\ell_2}, a_{k_1} | -a_{k_0})$.

(5) k_1 is incremented by 1, and then return to step (3) until $k_1 = M_1$. (6) k_0 is incremented by 1 and then return to step (3) until $k_0 = M$.

Assume that the starting probability distribution of response amplitudes at half-cycle zero is available. Half this distribution is denoted $p_{A_0}(a_k)$, $k = 1,\ldots,M$. Only half the distribution is represented since $a_k > 0$, $k = 1,\ldots,M$. The starting pmf, and all other pmf's, will be assumed symmetric about the zero amplitude, in what follows. In most cases this pmf will be zero for $k > M_1$, and generally

$$p_{A_0}(a_k) = \begin{cases} 0.5 , & k = 1 \\ 0 , & k \neq 1 \end{cases} \tag{42}$$

This is the zero start initial condition. Two options are available for definition of the starting pmf, and the formula given above represents one option. Half the probability can be accounted for, as it is in Equation 42. In this case, the marginal pmf's and joint pmf, to be computed later in this analysis, represent the chance that specific amplitude and energy dissipated combinations will be realized during a given half cycle. Only the probabilities for amplitude combinations that are negative at the start of the half cycle and positive at the end will be computed. Because of symmetry, the probabilities of amplitude combinations which have positive amplitudes at the beginning of the half cycle and negative amplitudes at the end can be found from those computed. Since the energy dissipated during a response half cycle depends only on the amplitude change, ΔE_j remains unchanged when the signs on the starting and ending amplitudes are both changed. (See Equations 33 and 35.) The second option for definition of the starting pmf accounts for all the probability by changing the value, 0.5 to 1.0 in Equation 42. In this case A_0 must be replaced by its absolute value, $|A_0|$, because of symmetry. Then, in later computations the marginal pmf's and joint pmf's involving $|A_j|$ are computed. As in the previous case, ΔE_j is not affected.

The joint pmf of ΔE_1, the increment of dissipated energy during the first half-cycle, and A_0 and A_1, the amplitudes at the beginning and end of the first half-cycle can be obtained using Equations 41 and 42.

$$p_{\Delta E_1 A_0 A_1}\left(e_{k_2}, -a_{k_0}, a_{k_1}\right) = p_{\Delta E_1 A_1 | A_0}\left(e_{k_2}, a_{k_1} | -a_{k_0}\right) p_{A_0}\ -a_{k_0}\ ,$$

$$k_0 = 1,\ldots,M_1$$
$$k_1 = 1,\ldots,M$$
$$k_2 = 1,\ldots,M' \qquad (43)$$

Take $p_{A_0}(-a_{k_0}) = p_{A_0}(a_{k_0})$, $k_0 = 1,\ldots,M_1$ because of the symmetry assumption. Only half the complete joint pmf is represented here, but because of the symmetry assumption the joint pmf evaluated at negative values, $-a_{k_0}$, is the mirror image of that evaluated in Equation 43. Specifically,

$$p_{\Delta E_1 A_0 A_1}\left(e_{k_2}, a_{k_0}, -a_{k_1}\right) = p_{\Delta E_1 A_0 A_1}\left(e_{k_2}, -a_{k_0}, a_{k_1}\right), \qquad \begin{aligned} k_0 &= 1,\ldots,M_1 \\ k_1 &= 1,\ldots,M \\ k_2 &= 1,\ldots,M' \end{aligned}$$

(43a)

The marginal pmf's of ΔE_1 and A_1 can be obtained by summing out dependence on k_0 and k_1, then k_0 and k_2, respectively.

$$\frac{1}{2} \cdot p_{\Delta E_1}\left(e_{k_2}\right) = \sum_{k_0=1}^{M_1} \sum_{k_1=1}^{M} p_{\Delta E_1 A_0 A_1}\left(e_{k_2}, e_{k_0}, e_{k_1}\right), \quad k_2 = 1,\ldots,M' \qquad (44)$$

$$p_{A_1}\left(a_{k_1}\right) = \sum_{k_0=1}^{M_1} \sum_{k_2=1}^{M'} p_{\Delta E_1, A_0 A_1}\left(e_{k_2}, e_{k_0}, e_{k_1}\right), \quad k_1 = 1,\ldots,M \qquad (45)$$

When Equation 45 is summed over k_1, the result is the value one-half. The reason is that half a symmetric pmf is represented. When Equation 44 is summed over k_2, the result is one-half. The reason is that, even though all the energy states are represented (see Equations 39a and b), only half the paths leading through these energy states are considered. This fact is accounted for by the factor of 1/2 on the left-hand side of Equation 44.

In the computation framework established here it is not feasible to allow the system to execute responses where yielding increases without limit. While the transition probabilities established in Equation 41 can account for this, the computer storage required to account for all the paths would be prohibitive. Therefore, a response collapsing procedure is established here. This procedure concentrates the probability associated with all those amplitudes a_k, $k > M_1$, into $p_{A_1}^{(m)}(a_{M_1})$. That is,

$$p_{A_1}^{(m)}(a_k) = \begin{cases} \sum_{k_0=M_1}^{M} p_{A_1}\left(a_{k_0}\right), & k = M_1 \\ p_{A_1}(a_k), & k < M_1 \end{cases} \tag{46}$$

The (m) superscript refers to the fact that the pmf of A_1 has been modified.

At this point the modification destroys the accuracy of the true amplitude pmf at half-cycle 1. Indeed, this change affects all future amplitude probability computations. However, the modification does not affect the accuracy of the increment in energy dissipated probability computations. The reason is this. Once a curve governing response in the SDF system spring-force-versus-displacement diagram has been identified, the only quantity which affects the energy dissipated during a half-cycle (besides the input and system parameters) is the starting amplitude on that curve. For example, consider Figure 11. If, at the end of a particular half-cycle of response, the system has reached the yield threshold, but has not yielded, then during the next half-cycle the response will be governed by curve No. 1. If, on the other hand, yielding occurs at the end of a half-cycle, then the next half-cycle may be governed by curve No. 2. If the system which yielded is excited by the same input as the system which did not yield, then the change in displacement will be the same in both systems, and both systems will dissipate the same amount of energy during their response half-cycles. This is shown in in the figure. The conclusion is that, when a system has yielded during a particular half-cycle of response, it is accurate, as far as energy dissipation calculations are concerned, to treat that system as though it started its next half-cycle of response from the yield displacement. This is precisely what the modification of Equation 46 does.

Now using Equations 41 and 46 the joint pmf of ΔE_2, A_1 and A_2 can be computed.

$$p_{\Delta E_2 A_2 A_1}\left(e_{k_2}, a_{k_1}, a_{k_0}\right) = p_{\Delta E_2 A_2 | A_1}\left(e_{k_2}, a_{k_1} | a_{k_0}\right) p_{A_1}^{(m)}\left(-a_{k_0}\right), \quad \begin{array}{l} k_0 = 1,\ldots,M_1 \\ k_1 = 1,\ldots,M \\ k_2 = 1,\ldots,M' \end{array} \tag{47}$$

This can be used to obtain the marginal pmf's of ΔE_2 and A_2. Then the marginal pmf of A_2 can be modified as in Equation 46. Then the joint pmf of ΔE_3, A_2 and A_3 can be obtained, and so on, until all the pmf's $p_{\Delta E_j}(e_k)$, $j = 1,\ldots,N'$, $k = 1,\ldots,M'$ are known.

Finally, the specification for the stochastic process E_j, $j = 0,\ldots,N'$, can be formed using the pmf's $p_{\Delta E_j}(e_k)$, $j = 1,\ldots,N'$, $k = 1,\ldots,M'$. E_j is the energy dissipated by the system through half-cycle j. E_j, $j = 0,\ldots,N'$,

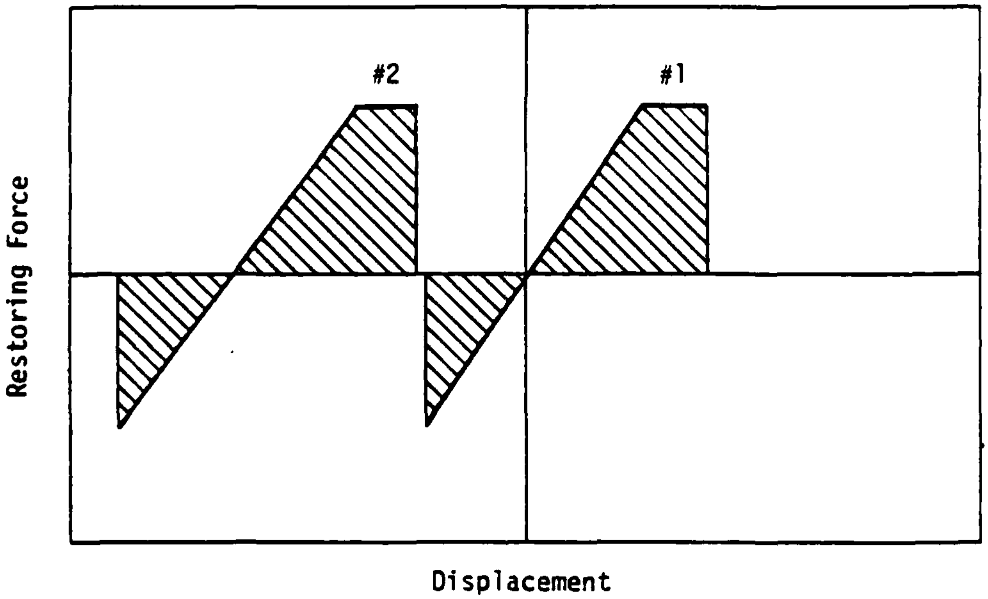

Figure 11. Force displacement response of elasto-plastic SDF systems.

is a Markov process. The energy dissipated by the system through nalf-cycle j depends only on the energy dissipated through half-cycle j-1 and the energy dissipated increment in the half-cycle between j-1 and j.

Let $p_{E_j|E_{j-1}}(\varepsilon_{k_1}|\varepsilon_{k_0})$, $j = 1,\ldots,N'$, k_0, $k_1 = 1,\ldots M''$, be the conditional pmf for the random process, E_j, $j = 0,\ldots N'$, defining transition probabilities between states of dissipated energy. Let

$$M'' = N' \times M' \tag{48}$$

and

$$\varepsilon_k = (k - \tfrac{1}{2})\ \Delta e\ , \qquad k = 1,\ldots,M'' \tag{49}$$

where Δe is defined in Equation 39a.

Then the transition probabilities are defined

$$p_{E_j|E_{j-1}}\left(\varepsilon_{k_1}|\,\varepsilon_{k_0}\right) = p_{\Delta E_j}\left(\varepsilon_{k_1} - \varepsilon_{k_0}\right), \qquad \begin{array}{l} j = 1,\ldots,N' \\ k_0,k_1 = 1,\ldots,M'' \end{array} \tag{50}$$

That is, the chance that the dissipated energy state changes from ε_{k_0} to ε_{k_1} in one half-cycle equals the chance that $e_{k_1} - e_{k_0}$ units of energy are dissipated during the jth half-cycle.

Let $p_{E_j}(\varepsilon_k)$, $j = 0,\ldots,N'$, $k = 1,\ldots,M''$, be the pmf of energy dissipated through the jth half-cycle. Then the joint pmf of energy dissipated through the j-1st and jth half-cycles is obtained following Equation 34a; it is

$$p_{E_{j-1}E_j}\left(\varepsilon_{k_0},\ \varepsilon_{k_1}\right) = p_{E_j|E_{j-1}}\left(\varepsilon_{k_1}|\,\varepsilon_{k_0}\right) p_{E_{j-1}}\left(\varepsilon_{k_0}\right), \qquad \begin{array}{l} j = 1,\ldots,N', \\ k_0,k_1 = 1,\ldots,M'' \end{array} \tag{51}$$

The marginal pmf of energy dissipated through the jth half-cycle is

$$p_{E_j}\left(\varepsilon_{k_1}\right) = \sum_{k_0=1}^{M''} p_{E_{j-1}E_j}\left(\varepsilon_{k_0},\ \varepsilon_{k_1}\right), \qquad \begin{array}{l} j = 1,\ldots,N' \\ k_1 = 1,\ldots,M'' \end{array} \tag{52}$$

Or by combining the operations in Equations 51 and 52, the marginal pmf of E_j from the pmf of E_{j-1} can be obtained

$$p_{E_j}\left(\epsilon_{k_1}\right) = \sum_{k_0=1}^{M''} p_{E_j|E_{j-1}}\left(\epsilon_{k_1}|\epsilon_{k_0}\right) p_{E_{j-1}}\left(\epsilon_{k_0}\right) , \qquad \begin{array}{l} j = 1,\ldots,N' \\ k_1 = 1,\ldots,M'' \end{array} \tag{53}$$

By specifying the starting pmf of E_0; one can obtain the pmf of E_j for any $j = 1,\ldots,N'$, through successive applications of Equation 53.

Note that no assumptions regarding the stationarity of the input signals has been made in the analysis of this section; therefore, the probability distribution of the energy dissipated response excited by a nonstationary input can be computed using the technique developed above. Further, energy dissipation caused by inelasticity in the spring and viscous damping need not be considered simultaneously. No feature in the theoretical development of this section precludes the consideration of energy dissipation due to one source only.

When the pmf, $p_{E_j}(\epsilon_{k_1})$, is summed over all k_1 the result will always be a value of 1. The reason is that all potential values the dissipated energy might assume have been approximately accounted for. The definition of e_{max} and e_{min} in Equations 38a and 38b, and use of these in Equation 39a and 39b, the definition of the dissipated energy states, has guaranteed this.

The accuracy of computations performed using the approach specified above will improve as the values of M, M_1, M' and a_M are increased. The first three values involve the fineness of the discretization; the last value is related to the displacement interval where response paths are accounted for.

b. Numerical example--This section gives a numerical example demonstrating use of a computer program which implements the analyses presented in Section

II.5.a. In this example the probability distribution of energy dissipated up to a particular time is computed as a function of time. Specifically, the pmf, $p_{E_j}(\epsilon_k)$, $j = 0,\ldots,N'$, $k = 1,\ldots M''$, is computed at the half-cycle points of an SDF system response. The input used to excite the system is a band-limited white noise. The system computation and input parameters are listed in Table 4.

TABLE 4. SYSTEM, COMPUTATION, AND INPUT PARAMETERS FOR ENERGY DISSIPATED pmf CALCULATION

$\Delta t = \pi/(5\omega_n)$	$\omega_n = 6.28$	
$\sigma_{\ddot{X}} = 1.0$		
$\zeta = 0.05$	$k = 39.5$	$D = 10.0$
$M = 18$		
$M_1 = 12$		
$M' = 20$		
$M'' = 1000$		
$a_M = 15.0$		
$N' = 50$		

The response pmf was computed at a time interval of ten half-cycles. The resulting pmf's were interpolated so that the results could be displayed as a sequence of pdf's. The results are shown in Figure 12. Each pdf is plotted in a vertical plane of pdf ordinate versus energy dissipated.

At time zero the system starts with no energy dissipated, so the first pdf is a delta function. As time progresses, the chances for increased amounts of dissipated energy increase, and the pdf's of E_j spread. The pdf's shown in Figure 12 resemble exponential pdf's, and simple computations show that the results are nearly exponentially distributed.

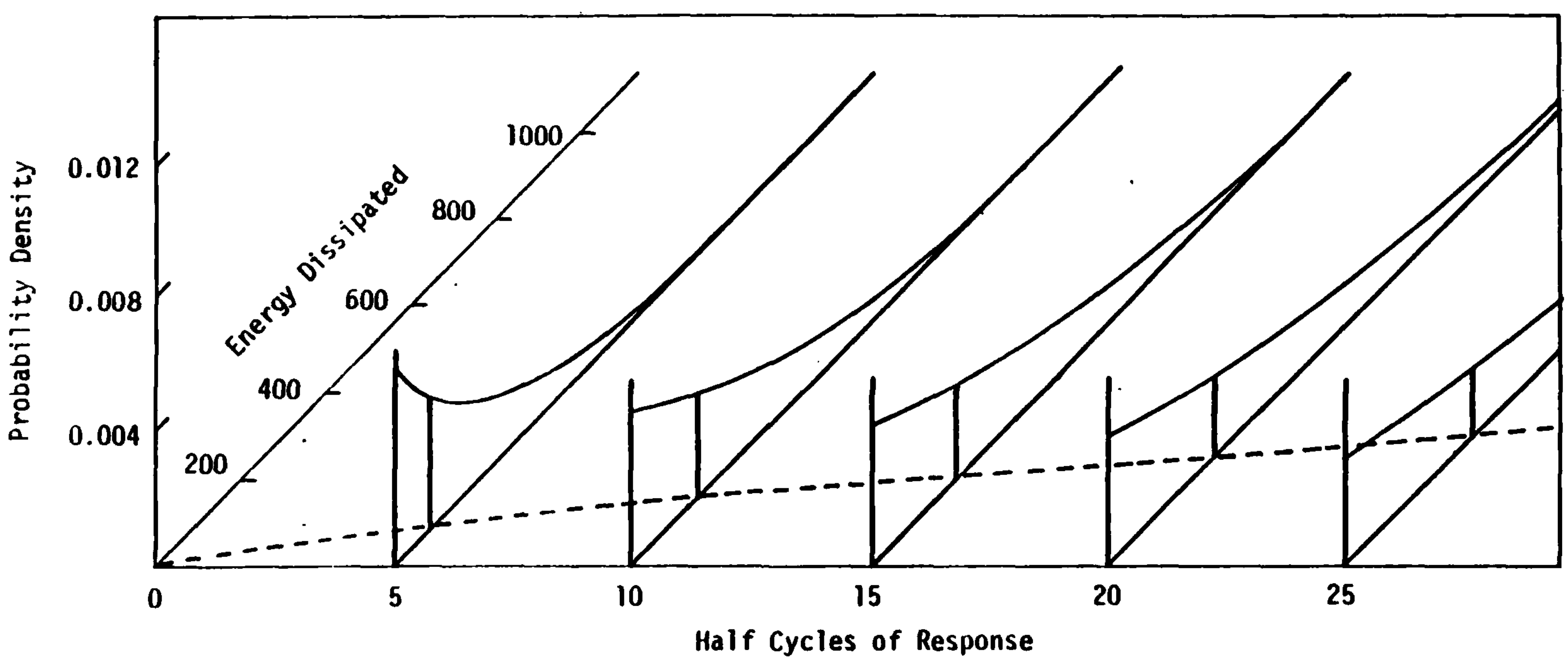

Figure 12. PDF of energy dissipated in an elasto-plastic SDF system as a function of time (ω_n = 6.28, ζ = 0.05, m = 1.0, σ_X = 39.5).

The mean and variance of each pdf were evaluated by the computer program. The graph of the mean value of energy dissipated versus time is plotted in the horizontal plane of Figure 12. This curve increases with time and appears to be in a condition of constant (straight line) increase past half-cycle 30.

A listing of the computer program used to perform this numerical example is given in the Appendix.

6. FIRST PASSAGE OF DISSIPATED ENERGY IN AN ELASTO-PLASTIC SYSTEM

a. Theoretical analysis--The analysis of Section II.5.a pointed out that the sum of the marginal pmf of dissipated energy, $p_{E_j}(\varepsilon_k)$, $j = 0,\ldots,N'$, $k = 1,\ldots,M''$, over all k equals 1, because all the values that dissipated energy can assume were accounted for. Also pointed out was that the stochastic process tracking dissipated energy is Markov. The first passage problem for energy dissipated in an elasto-plastic system can be solved by taking advantages of the Markov property and rearranging the computation scheme to permit the escape of some response paths past a barrier.

Since the stochastic process tracking dissipated energy is Markov, its transition probabilities can be arranged in a transition matrix defined as follows.

$$[P_{j+1}] = \begin{bmatrix} p_{E_{j+1}|E_j}(\varepsilon_1|\varepsilon_1) & p_{E_{j+1}|E_j}(\varepsilon_2|\varepsilon_1) & \cdots & p_{E_{j+1}|E_j}(\varepsilon_{M''}|\varepsilon_1) \\ p_{E_{j+1}|E_j}(\varepsilon_1|\varepsilon_2) & p_{E_{j+1}|E_j}(\varepsilon_2|\varepsilon_2) & \cdots & p_{E_{j+1}|E_j}(\varepsilon_{M''}|\varepsilon_2) \\ \vdots & & & \\ p_{E_{j+1}|E_j}(\varepsilon_1|\varepsilon_{M''}) & p_{E_{j+1}|E_j}(\varepsilon_2|\varepsilon_{M''}) & \cdots & p_{E_{j+1}|E_j}(\varepsilon_{M''}|\varepsilon_{M''}) \end{bmatrix},$$

$$j = 0,\ldots,N'-1 \qquad (54)$$

Each of the elements in the matrix is defined as in Equation 50. And when the pmf of dissipated energy is arranged in vector form,

$$\{p_j\} = \left(p_{E_j}(\epsilon_1)\ p_{E_j}(\epsilon_2)\ \cdots\ p_{E_j}(\epsilon_{M''})\right)^T , \qquad j = 0,\ldots,N' \tag{55}$$

Through specification of the pmf of dissipated energy at half-cycle zero, denoted $\{p_0\}$, one can obtain the pmf of dissipated energy at half-cycle j. This is

$$\{p_j\} = \prod_{k=1}^{j} [P_k]\ \{p_0\} \tag{56}$$

(Details of this general procedure are given in Section II.2.d.)

The computation procedure outlined above can be modified to yield first passage probabilities. Let $[P^*_{j+1}]$ denote a square submatrix of $[P_{j+1}]$ taken from the upper left corner. Then

$$[P^*_{j+1}] = \begin{bmatrix} p_{E_{j+1}|E_j}(\epsilon_1|\epsilon_1) & p_{E_{j+1}|E_j}(\epsilon_2|\epsilon_1) & \cdots & p_{E_{j+1}|E_j}(\epsilon_L|\epsilon_1) \\ p_{E_{j+1}|E_j}(\epsilon_1|\epsilon_2) & p_{E_{j+1}|E_j}(\epsilon_2|\epsilon_2) & \cdots & p_{E_{j+1}|E_j}(\epsilon_L|\epsilon_2) \\ \vdots & & & \\ p_{E_{j+1}|E_j}(\epsilon_1|\epsilon_L) & p_{E_{j+1}|E_j}(\epsilon_2|\epsilon_L) & \cdots & p_{E_{j+1}|E_j}(\epsilon_L|\epsilon_L) \end{bmatrix} ,$$

$$j = 0,\ldots,N'-1 \tag{57}$$

where $L \leq M''$. And let $\{p^*_0\}$ denote a vector of elements to be operated on by the $[P^*_j\}$. The elements in $\{p^*_0\}$ are the probabilities that E_0 occupies the states e_1 through e_L.

$$\{p^*_0\} = \left(p_{E_0}(\epsilon_1)\ p_{E_0}(\epsilon_2)\ \cdots\ p_{E_0}(\epsilon_L)\right)^T \tag{58}$$

The star (*) superscript is attached as an indicator that the transition and state probabilities corresponding to the entire range of realizations are not included in $[P^*_{j+1}]$ and $\{p^*_0\}$. A vector $\{p^*_j\}$ can be obtained through successive operations of $[P^*_k]$ on $\{p^*_0\}$. This is

$$\{p^*_j\} = \prod_{j=1}^{j} [P^*_k]\,\{p^*_0\}\ , \qquad j = 1,\ldots,N' \tag{59}$$

The elements of $\{p^*_j\}$ represent the probabilities that particular states in the energy dissipated range, ε_1 through ε_L, will be reached following paths that do not pass outside the level ε_L. Therefore, at half-cycle j, the no passage probability of dissipated energy is

$$P(T_1 > j\pi/\omega_n) = \sum_{k=1}^{L} p^*_{E_j}(\varepsilon_k)\ , \qquad j = 1,\ldots,N' \tag{60}$$

T_1 *is the random variable denoting the time at which first passage of the* dissipated energy response passes outside the level e_L. π/ω_n is the duration of a half-period of the system under consideration. $p^*_{E_j}(\varepsilon_k)$ is the kth element in the vector $\{p^*_j\}$.

The first passage event is the complement of the no passage event; therefore, the first passage probability is given by

$$P(T_1 \leq j\pi/\omega_n) = 1 - P(T_1 > j\pi/\omega_n)\ , \qquad j = 1,\ldots,N' \tag{61}$$

This first passage probability is a nondecreasing function of j.

As long as the input exciting response in Equation 31 has nonzero mean square power, the first passage probability will increase as a function of time. This corresponds to a feature in the matrix of Equation 57 where the rows do not add to 1; that is, the probability that the response starts at any point in the range ε_1 through ε_L and passes to another point in the same range is lower than 1. The remainder of the probability corresponds to paths which pass outside the barrier ε_L.

The analysis done in this section is restricted by the assumptions used in obtaining the elements in the transition matrix, Equation 54. Most important, the response is assumed narrow band; therefore, the analysis is accurate only for systems in which large amounts of yielding cannot occur during most half-cycles. First passage of energy dissipated due to the non-stationary response of an SDF system can be accounted for, though, since the probability distribution of energy dissipated in nonstationary response can be computed using the underlying approach of Section II.5.a.

b. <u>Numerical examples</u>--This section summarizes some numerical examples demonstrating the use of a computer program which implements the analysis of Section II.6.a. The computer program used here involves only slight modifications in the computer program of Section II.6.b; therefore, it is not listed in the Appendix. The first passage probability functions are computed for three cases. In each case the input excitation is a band-limited white noise stochastic process. Three barrier levels are chosen for analysis. The system, computation and input parameters are listed in Table 5.

The first passage probability was computed at an interval of every ten half-cycles. The first passage probability results were interpolated so that they could be displayed as continuous curves. These are shown in Figure 13. A zero start condition was used on each first passage computation; that is, the system was started with zero energy dissipated. For this reason, each curve starts at zero. As time progresses the chance that the response passes outside the barrier increases, so the first passage curve increases. As time goes to infinity these curves approach one since the input is stationary; excursion of any finite barrier is assured as time goes to infinity. As the

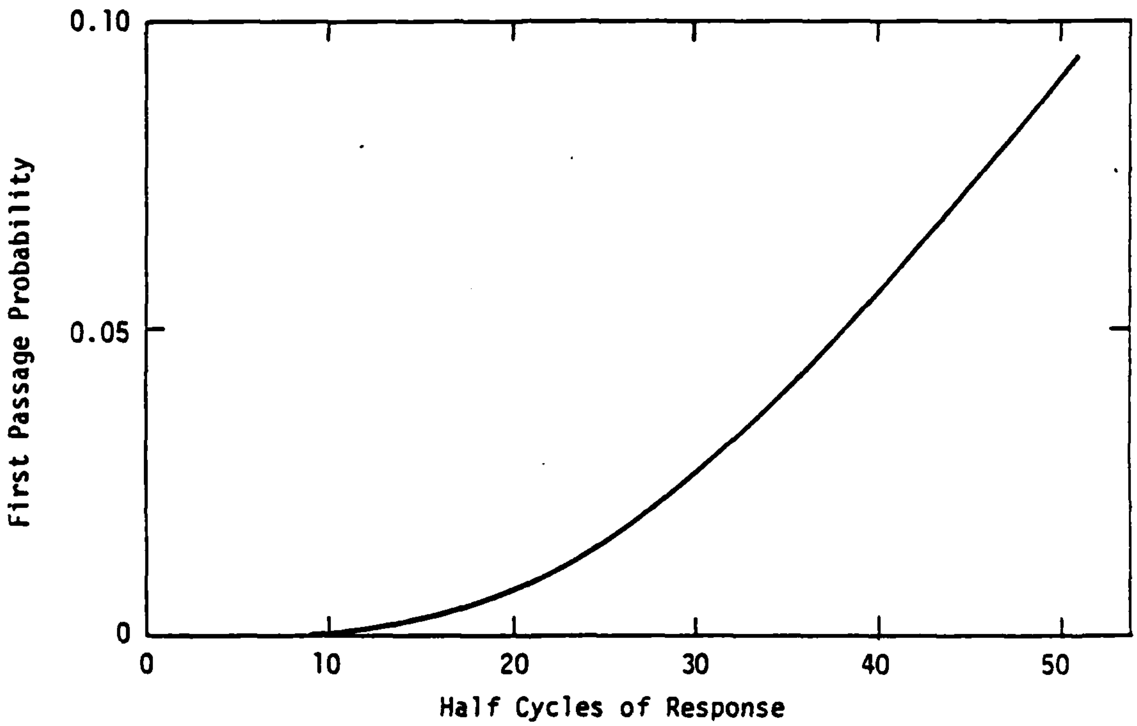

Figure 13. First passage probability of energy dissipated in an elasto-plastic SDF system [ω_n = 6.28, ζ = 0.05, m = 1.0, σ_X = 39.5, c = 1540 (energy barrier level)].

TABLE 5. SYSTEM, COMPUTATION, AND INPUT PARAMETERS FOR FIRST PASSAGE OF DISSIPATED ENERGY CALCULATION

$\Delta t = \pi/(5\omega_n)$	$\omega_n = 6.28$	
$\sigma_{\ddot{X}} = 1.0$		
$\zeta = 0.05$	$k = 39.5$	$D = 10.0$
$M = 18$		
$M_1 = 12$	$M' = 20$	
$a_M = 15.0$		
$N' = 50$		
Case 1	$L = 5$	$\epsilon_5 = 1400$
Case 2	$L = 10$	$\epsilon_{10} = 2800$
Case 3	$L = 15$	$\epsilon_{15} = 4200$

barrier height is increased the first passage probability tends to increase more slowly with time.

7. PROBABILITY OF FAILURE

a. Theoretical analysis--The reliability of a structure is the probability that it will perform satisfactorily over a preestablished period of time. The requirement for satisfactory performance implies that some criterion has been established to judge whether or not the response is acceptable. Failure of a system to satisfy the response criterion is called structural failure and the probability of failure is 1 minus the reliability. In many situations failure occurs only with the physical collapse of a structure; and in such cases the reliability is the chance that no collapse will occur during the design life of a structure. When structural collapse is the failure criterion the reliability is difficult to estimate accurately since a structural analysis predicting collapse must be executed in the reliability analysis. For this reason structural reliability is often simply bounded or estimated approximately, using approximate nonlinear structural analyses where necessary.

There are three features which may lead to randomness in the dynamic response of a structure. These are randomness in (1) the input, (2) the mechanical properties of the structural material, and (3) the geometry of the system under consideration. When the material properties and geometry of a structure are practically deterministic, then only the input is random; and the reliability analysis can be reduced, in many cases, to a first passage analysis like the one presented in Section II.2 or II.3 or II.6. The deterministic failure level of the system is simply chosen as the first passage barrier, and the failure probability is computed as the first passage probability.

In a more general situation the mechanical properties of the structural material are random and no single first passage analysis can be used to estimate the failure probability. Rather, a combination of information regarding the probabilistic character of the material must be used with information on the probabilistic character of the peak response to estimate the chance of failure. Specifically, through testing, the conditional probability of failure of a material can be developed. The probability of failure is conditional on some measure of the load on the material. For example, when a test specimen is loaded monotonically the probability of failure is conditioned on peak stress or peak strain. When a test specimen is loaded cyclically the probability of failure may be conditioned on a measure of accumulated plastic deformation, or energy dissipated, etc. The following material considers this problem for SDF systems. The response of an elasto-plastic system is considered, and the system under consideration is assumed to have a random capacity for dissipating energy. Through testing, the probability of failure can be specified, conditional upon the amount of energy dissipated in the system. When the letter F is used to denote the failure event, the conditional probability of failure is denoted $P(F|E_c = \varepsilon)$, $\varepsilon \geq 0$. E_c denotes the total energy dissipated over the entire duration of the system response, and the assumption is made that the conditional probability of failure is known for all realization of dissipated energy, ε. When the probability distribution of E_c is specified, then this can be used with $P(F|E_c = \varepsilon)$ to find the probability of failure.

The quantity E_c, whose probabilistic character needs to be specified, is the peak value of energy dissipated over the entire response duration. The cdf of this quantity can be found in terms of the first passage probability for dissipated energy obtained in Section II.6. The following material slightly changes the notation established in Equations 60 and 61. Let T_ε denote the random variable representing the time at which first passage of the dissipated energy barrier level ε occurs. If T_c is the duration of the response, then $P(T_\varepsilon \leq T_c)$ is the probability that first passage beyond the barrier ε occurs at or before T_c, and can be found from Equation 61.

But this probability can be viewed in another way. It is the chance that the peak value of energy dissipated in the time interval $(0, T_c)$ surpasses the value ε. That is

$$P(E_c > \varepsilon) = P(T_\varepsilon \leq T_c) \tag{62}$$

The complement of the event whose probability is described on the left-hand side is that the energy dissipated is less than or equal to ε; therefore,

$$P(E_c \leq \varepsilon) = 1 - P(T_\varepsilon \leq T_c) \tag{53}$$

The function on the left can be computed through repeated application of the first passage analysis. That is, a first passage analysis can be run through time T_c, for various values of the barrier level, ε; in this way $P(E_c \leq \varepsilon)$ is evaluated at a discrete set of values.

The pdf of E_c can be obtained by differentiating the cdf of E_c, Equation 63, with respect to ε. The result is

$$f_{E_c}(\varepsilon) = \frac{d}{d\varepsilon} P(E_c \leq \varepsilon) \ , \qquad \varepsilon \geq 0 \tag{64}$$

The product, $f_{E_c}(\varepsilon)\, d\varepsilon$, denotes the probability that E_c has a realization in the interval $(\varepsilon, \varepsilon + d\varepsilon]$; therefore, the joint probability that system failure occurs and the dissipated energy lies in the interval $(\varepsilon, \varepsilon + d\varepsilon]$ is given by

$$P(F, \ \varepsilon < E_c \leq \varepsilon + d\varepsilon) = P(F | E_c = \varepsilon)\, f_{E_c}(\varepsilon)\, d\varepsilon \ , \qquad \varepsilon \geq 0 \tag{65}$$

The probability of failure can be obtained by integrating out dependence on ε in the above expression. The result is

$$P(F) = \int_0^\infty P(F | E_c = \varepsilon)\, f_{E_c}(\varepsilon)\, d\varepsilon \tag{66}$$

The above expression can be discretized so that the discrete nature of the cdf of E_c can be directly accounted for.

The computations performed above assume that the SDF system will fail due to the dissipation of energy in the system. In fact, failure may be more strongly related to another measure of structural response or a collection of other measures. If the conditional probability of failure given these other measures of response can be specified, either based on experiments or theoretical considerations, and if the probabilistic character of these other measures of structural response can be obtained, based on the type of analysis given in this report, then the probability of failure can be specified using the same general approach developed in this section. In view of this, the failure analysis presented in this section is simply meant to demonstrate a technique which takes into account the randomness in the excitation and the system characteristics.

b. Numerical example--This section presents a numerical example demonstrating the use of a computer program which implements the failure analysis of Section II.7.a. The computer program uses the first passage analysis of Section II.6.b to develop the peak response cdf; therefore, the input used in that section is in effect here. That is, a white noise is used to excite the system. The time duration of the input and response are taken to be 50 half-cycles of the response. The cdf and pdf of E_c are evaluated at that time. The system, computation, and input parameter used to obtain the cdf of E_c are given in Table 6.

TABLE 6. SYSTEM, COMPUTATION, AND INPUT PARAMETERS FOR PROBABILITY OF FAILURE CALCULATION

$\Delta t = \pi/(5\omega_n)$		
$\sigma_{\ddot{X}} = 1.0$		
$\zeta = 0.05$	$k = 39.5$	$D = 10.0$
$M = 18$	$M_1 = 12$	$M' = 20$
$a_M = 15.0$		
$N' = 50$		
cdf of E_c evaluated at		
$\varepsilon_3 = 840$	$\Delta E = 280$	
$\varepsilon_4 = 1120$		
$\varepsilon_5 = 1400$		
$\varepsilon_6 = 1680$		
$\varepsilon_7 = 1960$		

Part of the cdf of E_c was computed at increments graphed in Figure 14. The cdf was computed at increments of dissipated energy equal to 280. The cdf is shown as a step function. The pdf is estimated by differencing the cdf. The increments obtained by differencing the cdf are shown in Figure 15.

In order to compute the failure probability, the conditional probability of failure given the level of dissipated energy must be known. In this example it is assumed that the conditional probability of failure is that shown in the graph of Figure 16. The dissipated energy increments upon which the failure probability is conditioned were chosen to correspond to those used in obtaining the cdf of E_c. This was done for convenience, though any discretization in the conditional failure probability function could be interpolated to make it compatible with the increments in the cdf of E_c.

The discretized form of Equation 66 used to obtain the failure probability is

$$P(F) = \frac{1}{\Delta E} \sum_{k=1}^{M_2} P(F \mid E_c = \tfrac{1}{2}(\varepsilon_k + \varepsilon_{k+1}))\, [P(E_c < \varepsilon_{k+1}) - P(E_c < \varepsilon_k)] \qquad (67)$$

where ΔE is the dissipated energy increment ε_k, $k = 1,\ldots,M_2$ are the dissipated energy levels where the cdf of E_c is computed, and M_2 is the number of dissipated energy values where the computation is carried out. In the present example, the probability of failure assumes the value

$$\begin{aligned} P(F) &= 0.0(0.6057) + 0.2(0.1979) + 0.5(0.1031) + 0.8(0.0532) + 1.0(0.0401) \\ &= 0.1750 \end{aligned} \qquad (68)$$

This example shows that it is necessary to obtain the cdf of E_c over these values only where the conditional probability of failure is greater than zero and less than 1. None of the energy dissipated quantities can lead to failure when the conditional probability of failure is zero, and all the energy dissipated quantities lead to failure when the conditional probability of failure is 1.

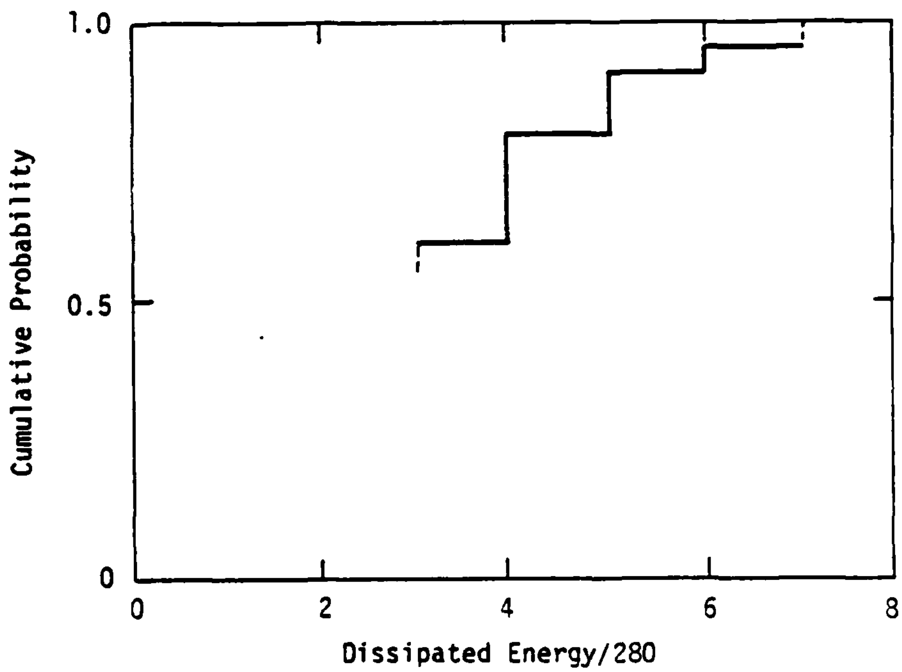

Figure 14. Partial cdf of energy dissipated during 50 half cycles of response of an elasto-plastic SDF system.

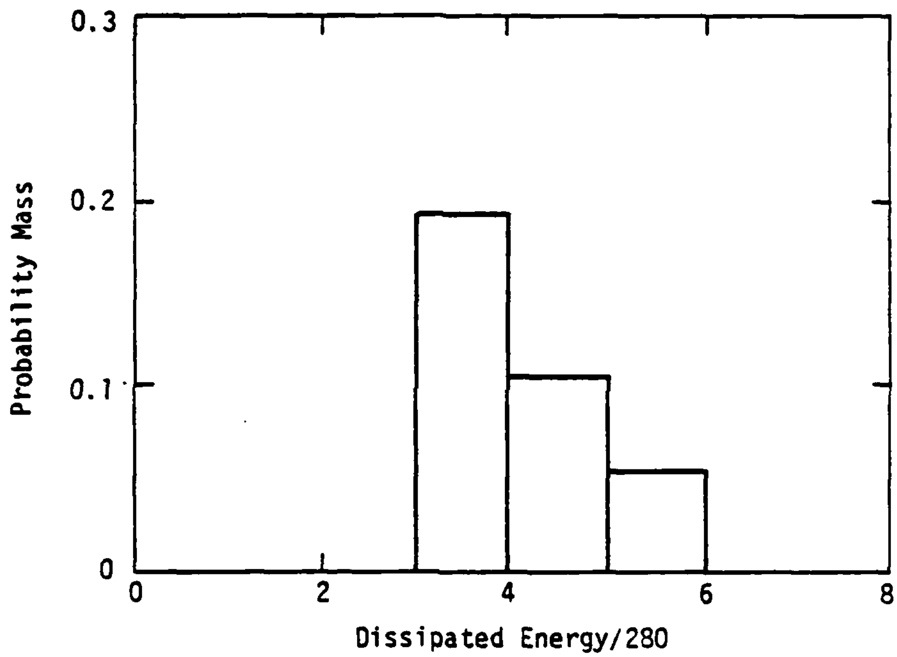

Figure 15. Approximate partial pdf of energy dissipated during 50 half cycles of response of an elasto-plastic SDF system.

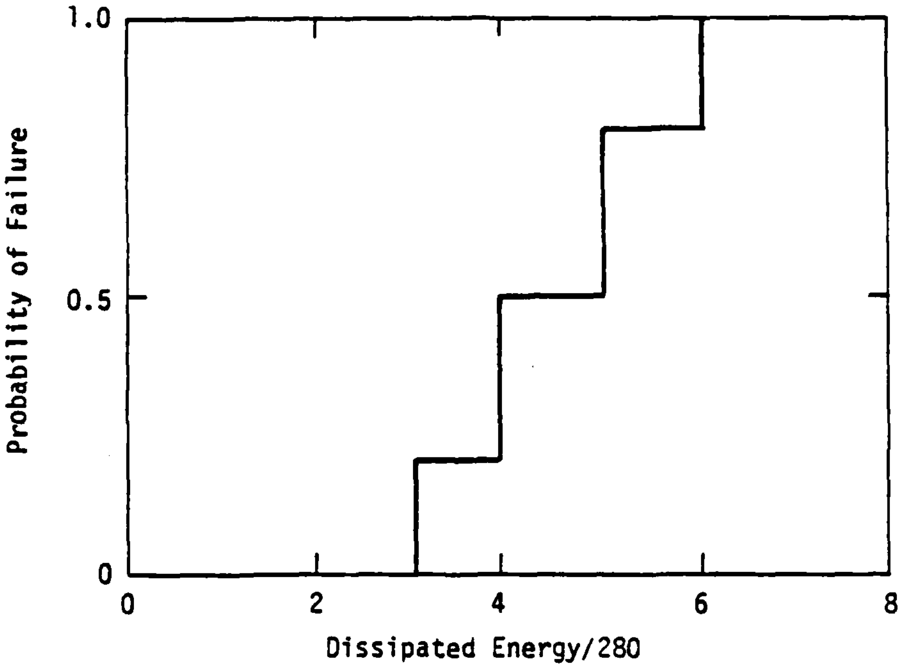

Figure 16. Conditional probability of failure of an elasto-plastic SDF system given dissipated energy.

In summary, this section has shown an approach to the prediction of reliability of SDF systems that are nonlinear and subjected to both stationary and nonstationary loads. Various measures of response have been considered and the failure probability has been computed for a system with a random failure level of dissipated energy.

III. FAILURE ANALYSIS OF MULTIPLE-DEGREE-OF-FREEDOM SYSTEMS

1. INTRODUCTION

In the previous section, a probabilistic analysis of single-degree-of-freedom (SDF) systems was presented. The extension of the problem to multiple-degree-of-freedom (MDF) systems is not trivial. Recall that one of the disadvantages of the previously presented SDF approach was the storage requirements for the state probabilities and the transition probabilities. Even when some memory requirements were reduced by additional computational effort, the memory storage was large. Any solution for MDF systems should include considerations for a reasonably large number of degrees of freedom (at least several hundred). This constraint has led to an alternate procedure for predicting the probability of survival of structures excited by highly transient loads, such as blast and shock.

The survival probability of a structural system subjected to blast and shock loads can be computed using the basic elements described in the opening section of this report. The random character of the structural system and of the loadings must be known. A mathematical procedure must be developed that models the system so that the random response is predicted. The failure level must be characterized and, finally, intersection of response and failure must be integrated into a probability of failure or its complement, the probability of survival. This section presents an approach using these elements for inelastic systems that are represented by MDF solutions that have some chance of failing at any of a number of locations.

2. THEORETICAL ANALYSIS

a. Failure at a single point--The survivable probability of a blast-excited structure can be computed by a four-step procedure: (1) A numerical scheme for computing the response at various locations on an inelastic structure is selected. (2) The parameters of the input and the structure are characterized probabilistically. (3) Peak responses are expanded in a series involving the input and system parameters, and the mean and variance of the peak response at each location are determined. (4) Margins of survival are established and used to compute the probability of survival. Each of these steps will now be described in detail.

The equation governing the response of an MDF structure can be written

$$[m]\{\ddot{z}\} + \{R(z)\} = \{F(t)\} \tag{69}$$

where $[m]$ is the mass matrix, $\{z\}$ is the displacement vector, dots indicate differentiation with respect to time, $\{R(z)\}$ is the restoring force vector, and $\{F(t)\}$ is the forcing vector. When the system characteristics are random, the parameters governing the behavior of $\{R(z)\}$ are random variables. When the input is a random process, the parameters of the forcing vector are random variables.

Let β_j, $j=1, \cdots, m$, denote the random structural parameters. Let β_j, $j=m+1, \cdots, n$, denote the random input parameters, then the response at a point, i, on the structure is a random process and can be expressed $z_i = z_i\,(t, \beta_1, \cdots, \beta_n)$. A number of techniques can be used to compute this response when the β_j, $j=1, \cdots, n$, are specified. In reality most dynamic problems do not explicitly solve the system of equations shown in Equation 69, but rather the system is spatially discretized in some way and the equations of motion are solved on a cell-by-cell basis in a sweep through the system for a small time step. Simplified approaches can be used as well as complex finite element solutions.

The maximum response in time at the i^{th} point is

$$Z_i(\beta_1, \cdots, \beta_n) = \max_t \left| z_i(t, \beta_1, \cdots, \beta_n) \right| \tag{70}$$

Note that although Z_i and z_i represent displacement here, they could represent any measure of the response and its maximum. Let μ_j and σ_j^2 represent the mean and variance of the random variable β_j, $j=1, \cdots, n$, and let ρ_{jk} be the correlation coefficient for β_j and β_k. The function Z can be expanded in a Taylor series about the means of the parameters.

$$Z_i = Z_i \Big|_{\{\mu\}} + \sum_{j=1}^{n} \frac{\partial Z_i}{\partial \beta_j} \Big|_{\{\mu\}} (\beta_j - \mu_j) + \cdots \tag{71}$$

where $\{\mu\}$ is the vector of mean values of the β_j, $j=1, \cdots, n$.

The mean and variance of the peak response can be computed approximately as

$$E[Z_i] \cong Z_i \Big|_{\{\mu\}} \tag{72a}$$

$$V[Z_i] \cong \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial Z_i}{\partial \beta_j} \Big|_{\{\mu\}} \frac{\partial Z_i}{\partial \beta_k} \Big|_{\{\mu\}} \rho_{jk} \, \sigma_j \, \sigma_k \tag{72b}$$

These expressions are approximate because higher order terms in the series expansion have been neglected. These terms can be retained when necessary. Several appropriate evaluations of the importance of the higher order terms can be considered. If higher order terms do not significantly change the solutions, they need not be retained. Also if the peak response function can be easily differentiated, either analytically or numerically, the slope, curvature or higher order differentials can be evaluated on either side of the mean to detect changes. If the changes are zero or small, no further terms should be necessary. An equivalent approach is to examine the order of the peak response function in a reasonable range of the mean determined by the failure level.

Structural failure is assumed to occur when the peak response at any single point exceeds the failure level. Let L_i be the random variable denoting the failure level at a point i. Let $E[L_i]$ and $V[L_i]$ be the mean and variance of L_i. Assume that Z_i and L_i are independent. A margin of survival at point i, M_i, can then be established as

$$M_i = L_i - Z_i \tag{73}$$

This margin has mean

$$E[M_i] = E[L_i] - E[Z_i] \tag{74}$$

and variance

$$V[M_i] = V[L_i] + V[Z_i] \tag{75}$$

Failure occurs when the margin of survival is negative and, therefore, the probability of survival is

$$P_S = P(M_i > 0) \tag{76}$$

This can usually be evaluated using $E[M_i]$ and $V[M_i]$ when the distribution of M_i is specified. When Z_i and L_i are either normally or lognormally distributed, P_S can easily be determined.

A development analogous to the above can be executed when Z is any measure of the structural response.

The following numerical example considers the case where the potential for failure exists at only one point on a structure.

b. <u>Numerical example</u>--A laterally supported vertical cylinder composed of two materials can be simply modeled as a set of axial springs. The two materials are modeled as springs in parallel, while the structure length is modeled as springs in series. An example structure and model are shown in Figure 17. The materials are assumed elasto-plastic.

Consider the structure excited by a random blast input defined by $P = A + Be^{-\alpha t}$ where P is the pressure history. The peak pressure A' = A+B is a normally distributed random variable with mean 1280 psi (8.8 MPa) and standard deviation 102 psi (0.7 MPa), B = -1.069A, and α is a normally distributed random variable with mean 0.80 and standard deviation 0.2 t as time.

The peak response, taken here as strain, at point C (Fig. 17) has been predicted for the mean values of the input using a numerical approximation to the solution of Equation 69 (Ref. 53). Additionally the peak responses near the means provide partial differential quotients to approximate the partial differentials in Equation 72b. The results obtained are

$$Z\big|_{\{\mu\}} = 2.12\ (10^{-3});$$

$$\partial Z/\partial A\big|_{\{\mu\}} = 0.429\ (10^{-3});$$

$$\partial Z/\partial \alpha\big|_{\{\mu\}} = -0.300\ (10^{-3});$$

and $\rho_{\alpha A}$ is assumed 0.1.

Using Equations 72a and b, the mean and variance of Z become 2.12 (10^{-3}) and 9.0 (10^{-8}). Let L be the strain at which failure occurs and define the mean of L as 0.002 and 2.5 (10^{-7}), respectively. From the above development, the mean and variance of M, the margin of survival, become μ_m = 1.2 (10^{-4}) and V_m = 3.4 (10^{-7}).

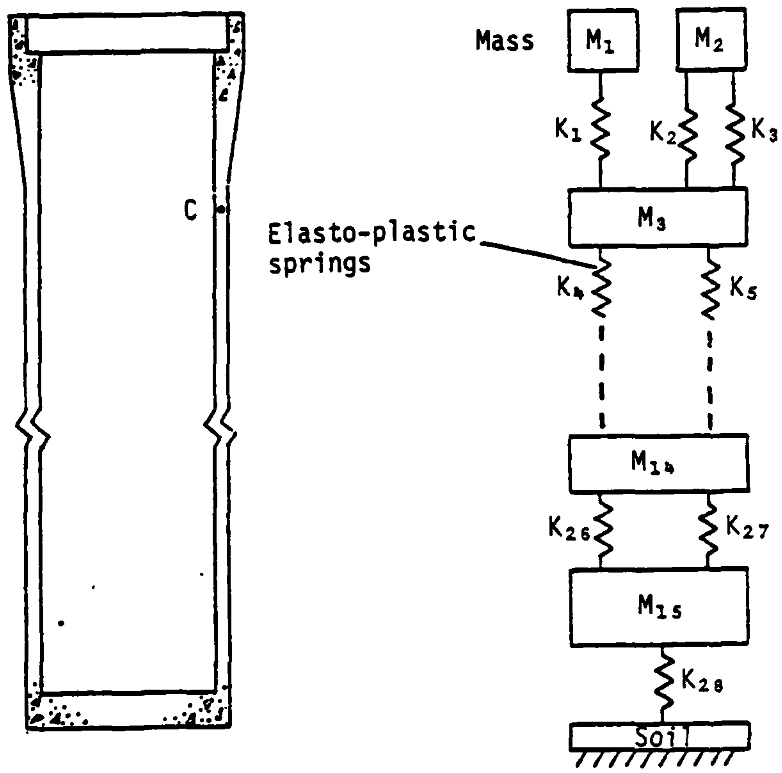

Figure 17. Structure with spring mass model.

Assuming a normal distribution for Z and L gives the probability of survival as

$$Ps = 1 - \phi \frac{1.2\ (10^{-4})}{5.8\ (10^{-4})} = 41.8\%$$

where ϕ is the standard normal cumulative distribution function.

In the present application, sufficient accuracy was obtained using the first two terms in the Taylor series (Equation 71). More complicated situations may require inclusion of additional terms.

c. Failure at multiple points--Note the possibility of multiple point failure. That is, the probability of structural failure, given that it could conceivably fail at any of a number of locations, N. If the failure level at one location is independent of failure at other locations, then the solution is simple. If, however, the failure level at one location is correlated to failure at other locations, the correlation coefficient for the failure levels must be given and the correlation for the margin of survival between locations must be determined.

The correlation coefficient for the margin of survival at any two locations, ℓ and m, can be computed:

$$\rho_{\ell,m}^{M} = \frac{\text{Cov}\ [M_\ell,\ M_m]}{\sqrt{V[M_\ell]\ V[M_m]}} \tag{77}$$

Note that $\text{Cov}[M_\ell,\ M_m] = E[M_\ell\ M_m] - E[M_\ell]\ E[M_m]$

$$= E[(Z_\ell - L_\ell)(Z_m - L_m)] - E[M_\ell]E[M_m]$$

$$= E[Z_\ell Z_m] + E[L_\ell L_m] - E[Z_\ell L_m] - E[L_\ell Z_m] - E[M_\ell][M_m]$$

now

$$E[L_\ell L_m] = \rho^L_{\ell,m}\sqrt{V[L_\ell]\,V[L_m]} + E[L_\ell]\,E[L_m]$$

where

$\rho^L_{\ell,m}$ is the specified correlation coefficient between the failure levels at locations ℓ or m

and

$$E[Z_\ell L_m] = E[Z_\ell]\,E[L_m]$$

$$E[L_\ell Z_m] = E[L_\ell]\,E[Z_m] \text{ by independence.}$$

Combining the above gives:

$$= Z_\ell(\mu)\,Z_m(\mu) + \sum_{j=1}^{n}\sum_{k=1}^{n} \rho_{jk}\,\sigma_j\,\sigma_k \left.\frac{\partial Z_\ell}{\partial \beta_k}\right|_{\{\mu\}} \left.\frac{\partial Z_m}{\partial \beta_j}\right|_{\{\mu\}}$$

$$+ \rho^L_{\ell,m}\sqrt{V[L_\ell]\,V[L_m]} + E[L_\ell]\,E[L_m] - E[Z_\ell]\,E[L_m]$$

$$- E[Z_m]\,E[L_\ell] - E[M_\ell]\,E[M_m] \quad \sqrt{V[M_\ell]\,V[M_m]} \tag{78}$$

which contains only previously determined or specified elements.

Failure occurs when the margin of survival is negative, and the probability of failure can therefore be approximated by

$$P_f = P\left(\bigcup_{i=1}^{N} M_i < 0\right) = \sum_{i=1}^{N} P(M_i < 0) - \sum\sum_{i \neq j} P(M_i < 0 \quad M_j < 0)$$

$$+ \cdots \pm P\left(\sum_{i=1}^{N} M_i < 0\right) \tag{79}$$

where N is the number of failure points considered.

This series may be approximated by truncating the series of summations after several terms. The problem then becomes

$$P\left(\bigcap_{i\varepsilon s} M_i < 0\right) \tag{80}$$

where s is a subset of the set $\{1, 2, \cdots, N\}$. This expression can be evaluated by executing the n-fold integral of the n^{th} order normal pdf of X_j, $j=1, \ldots, n$. (Assume that the set s contains n components and these are denoted X_j, $j=1, \cdots, n$.)

The n^{th} order normal joint pdf is

$$p_{M_1, \ldots, M_n}(x_1, \cdots, x_n) = \frac{1}{(2\pi)^{n/2}\,|S|^{1/2}}$$

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-\mu_i)\ r_{ij}\ (x_j-\mu_j)\right)$$

$$-\infty < x_i < \infty,\ i=1, \cdots, n \tag{81}$$

where $|S|$ is the covariance matrix of the X_i, $j=1, \cdots, n$, $|S|$ is the determinant of $|S|$, r_{ij} is the cofactor of the i^{th} row, j^{th} column element in $|S|$.

In terms of Equation 81, the expression 80 is written

$$P\left(\bigcap_{i\varepsilon s} M_i < 0\right) = \int_{-\infty}^{0} dm_1, \cdots \int_{-\infty}^{0} dm_n\ p_{M_1, \ldots, M_n}(m_1, \cdots, m_n)$$

$$= \int_{-\infty}^{0} dm_1, \cdots \int_{-\infty}^{0} dm_n\ \frac{1}{(2\pi)^{n/2}\,|S|^{1/2}}$$

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-\mu_i)\ r_{ij}\ (x_j-\mu_j)\right) \tag{82}$$

The integral can be modified through a change of variables

$$y_i = \frac{m_i - \mu_i}{\sigma_i}, \quad i=1, \cdots, n \tag{83}$$

to obtain

$$P\left(\bigcap_{i \epsilon s} M_i < 0\right) = \int_{-\infty}^{-\mu_1/\sigma_1} dy_1, \cdots \int_{-\infty}^{-\mu_n/\sigma_n} dy_n \frac{\sigma_1, \cdots, \sigma_n}{(2\pi)^{n/2} \ |S|^{1/2}}$$

$$\exp\left(-\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i y_i \ r_{ij} \ \sigma_j r_j\right) \tag{84}$$

The integral can be further modified by another change of variables

$$z_i = \frac{1}{y_i}, \quad i=1, \cdots, n \tag{85}$$

This yields

$$P\left(\bigcap_{i \epsilon s} M_i < 0\right) = \int_{-\sigma_1/\mu_1}^{0} dz, \cdots \int_{-\sigma_n/\mu_n}^{0} dz_n \frac{\sigma_1 \cdots \sigma_n}{(2\pi)^{n/2} \ |S|^{1/2}} \frac{1}{z_1^2 \cdots z_n^2}$$

$$\exp\left(-\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\sigma_i \ r_{ij} \ \sigma_j}{z_i \ z_j}\right) \tag{86}$$

This quantity can be evaluatec numerically.

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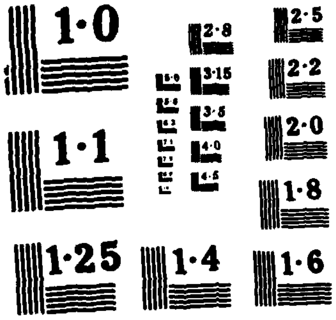

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In the present investigation, a Simpson's rule scheme is used to numerically evaluate the integral. Each of the integrals evaluated in Equation 86 is an element in one of the sums on the right side of Equation 79.

Thus, using $E[M_i]$, $V[M_i]$, and $\rho_{\ell,m}^{M}$, the probability of failure can be obtained.

d. Numerical example--A nine degree-of-freedom spring mass system that models a structure responding only axially has been selected to illustrate the analytical procedure (Fig. 18).

Note: All masses are 1000 kg.

Figure 18. Spring mass system.

The system has elasto-plastic springs. Two of the springs near the center (springs 5 and 6) exhibit yield at their mean values at lower levels than do the others. The forcing function is exerted on the first mass as shown in Figure 18 and is a decaying exponential of the form $F = A + Be^{-\alpha t}$. Three of the parameters that describe the system and two that describe the applied force were identified as random. The random variable parameters (mean and variance) and the correlation coefficients between sets of random variables are shown in Table 7.

The problem was constructed so that at the mean values of the parameters, the maximum strain in the system would result in a ductility ratio of about 10.

The partial quotients (that estimate the partial derivatives of the response measure with respect to each of the random parameters) were obtained by a series of deterministic calculations in which the random parameters were varied. The deterministic approach was the numerical solution used in the previous example.

TABLE 7. RANDOM PARAMETERS

Random Variables	Mean	Variance	Correlation Coefficients C_1	C_2	A+B	α
Spring constant, all springs, K	1.00E10	1.00E18	0.1	0.1	0	0
Yield limit, springs 1-4, 7-9, C_1	1.00E08	1.00E14		0.85	0	0
Yield limits, springs 5, 6, C_2	8.50E07	7.23E13			0	0
Peak applied force, A+B	1.00E08	1.00E14				0.9
Decay coefficient, α	15	2.25				

Five locations were selected as potential failure points; one near the loading (spring 2), two at the low-yield springs (springs 5 and 6), and two near the bottom (springs 8 and 9). The failure level at each point was considered as random; the means and the variances are shown in Table 8. The correlation coefficients between the failure levels at the various points were also estimated and are shown in the table.

TABLE 8. FAILURE LEVEL MEAN AND VARIANCE

Location (Spring Number)	Mean	Variance	Correlation Coefficients 5	6	8	9
2	0.06	3.6E-05	0.1	0.1	0.3	0.3
5	0.09	8.1E-05		0.8	0.1	0.1
6	0.09	8.1E-05			0.1	0.1
8	0.06	3.6E-05				0.7
9	0.06	3.6E-05				

Equations 74 and 75 have been used to estimate the mean and variance of the margin of survival at each location. Equation 78 estimates the correlation coefficients between the margins of survival at the various points. This information has been used as input in an approximation of the multivariate normal distribution, and a numerical form of Equation 79 has been used to compute the probability of system failure. The analysis shows a failure probability of 56.4 percent.

In summary, this section has shown the development of an approach that predicts the reliability of MUF systems that can fail at any of a number of locations. The random nature of the structural system and the load have been addressed. In addition, the failure level has been considered as a random property. The development is general in that the random parameters are treated as correlated rather than independent.

IV. DEVELOPMENT OF A STOCHASTIC FINITE ELEMENT CODE

1. INTRODUCTION

The previous section addressed the use of a deterministic finite element code to calculate the mean and to approximate the variance of selected response measures for nonlinear problems with random input and material properties. The approach required repetitive calculations to estimate the mean and the behavior near the mean so that the variance could be estimated. Since nonlinear, dynamic problems are random processes it would be more useful to have an approach that tracks the evolution of the random behavior of response measures with time. This section addresses such an approach. The equations in a typical numerical approach may be used to compute the variance and covariance of various response parameters; but this section carries the results only as far as the random description of the response measures. The extension for probabilistic failure analysis is the same as for the previous section.

2. THEORETICAL DEVELOPMENT

Consider a one-dimensional nonlinear finite element algorithm which proceeds as follows. The acceleration at location i and at the jth time step is

$$a_j^i = \left(\frac{F_j^i - FI_{j-1}^i}{m^i} \right)$$

where

a = acceleration

F = applied force

FI = internal force

m = mass

superscript i is a spatial index

subscript j is a time index

The velocity, V, at location i, time step j, for a Δt time step is

$$V_j^i = V_{j-1}^i + a_j^i \Delta t$$

and the displacement, d, becomes

$$d_j^i = d_{j-1}^i + V_j^i \Delta t$$

Using a simple definition of strain based on the original length, ℓ, the strain ε becomes

$$\varepsilon_j^i = \frac{d_j^{i+1} - d_j^i}{\ell}$$

and stress, σ, is addressed in some nonlinear, inelastic relationship to strain

$$\sigma_j^i = f\left(\varepsilon_j^i, \varepsilon_{m_{j-1}}^{i}, \sigma_{m_{j-1}}^{i}, E, ..\right)$$

where $\varepsilon_{m_{j-1}}^{i}$ and $\sigma_{m_{j-1}}^{i}$ are memory parameters (from the last time step) that are usually required for path dependent models. E and other parameters are the moduli and strains that are used in the functional form of the nonliner model.

An example of a nonlinear model (used in subsequent calculations in this report) is a two-modulus elastic-plastic model that unloads on the elastic modulus. For strains less than some specified strain ε_1, the material model is elastic with modulus E, that is

$$\sigma_j^i = E\varepsilon_j^i$$

when strains lie in the range $\varepsilon_1 < \varepsilon_j^i < \varepsilon_2$, then a second modulus, E2, becomes effective and

$$\sigma_j^i = E\varepsilon_1 + \left(\varepsilon_j^i - \varepsilon_1\right) E2$$

and when the strain exceeds $\varepsilon 2$ the material is plastic with

$$\sigma_j^i = E\varepsilon_1 + (\varepsilon_1 - \varepsilon_2) E2$$

This particular model can treat unloading differently than loading and thus dissipate energy. This treatment requires memory parameters. The memory parameters are the maximum stress and strain attained, so that at unload the stress can be computed from

$$\sigma_j^i = \sigma_{max_j}^{\ i} - \left(\varepsilon_{max_j}^{\ i} - \varepsilon_j^i\right) E$$

The last computation for a time step in this finite element process is the calculation of the internal force FI for the next time step.

$$FI_j^i = \left(\sigma_j^{i-1} - \sigma_j^i\right)A$$

where

A = cross section area of the element

$\sigma_j^0 = 0$

After all elements, i = 1. . .N, and nodes, i = 1---N + 1, are updated, the information generated at j is used to compute the response at j + 1.

The fact that all the memory required for calculation of the response measures for the current time step lies in the previous time step information indicates that this may be a Markov Process. A Markov Process is one in whicn the properties of a random process can be computed given the state at a previous time and the time since that state. The random properties of each of the response measure parameters can be computed from the previous time step data. For example the first output measure of interest that is computed by the deterministic algorithm is the velocity

$$V_j^i = V_{j-1}^i + a_j^i \Delta t$$

but

$$a_j^i = \left(\frac{F_j^i - FI_{j-1}^i}{m^i}\right)$$

so that

$$V_j^i = V_{j-1}^i + \left(F_j^i - FI_{j-1}^i\right) \frac{\Delta t}{m^i}$$

Assuming that V^i_{j-1} and FI^i_{j-1} are random variables, the properties of which were computed last time step and that F^i_j is a random variable, the properties of which describe the loading function, then

$$E\left[V^i_{j-1}\right],\ Var\left[V^i_{j-1}\right],\ E\left[FI_{j-1}\right],\ Var\left[FI_{j-1}\right],\ E\left[F^i_j\right],\ Var\left[F^i_j\right],$$

and

$$Cov\left[V^i_{j-1}FI_{j-1}\right]$$

are known. Then

$$E\ V^i_j = E\left[V^i_{j-1} + \left(F^i_j - FI^i_{j-1}\right)\frac{\Delta t}{m}\right]$$

= deterministic calculation with mean values

$$Var\left[V^i_j\right] = E\left[V^{i^2}_j\right] - E\left[V^i_j\right]^2$$

$$E\left[V^{i^2}_j\right] = E\left[V^i_{j-1} + \left(F^i_j - FI^i_{j-1}\right)\frac{\Delta t}{m}\right]^2$$

$$= E\left[V^{i^2}_{j-1} + 2\,\frac{\Delta t^2}{m}\left(V^i_{j-1}F^i_j - V^i_{j-1}FI^i_{j-1}\right)\right]$$

$$+ \frac{\Delta t^4}{m^{i2}}\left(F^{i^2}_j - 2F^i_jFI^i_{j-1} = FI^{i^2}_{j-1}\right)$$

so that

$$Var\left[V_j^i\right] = Var\left[V_{j-1}^i\right] - 2\frac{\Delta t^2}{m^i} Cov\left[V_{j-1}^i FI_{j-1}^i\right] + \frac{\Delta t^4}{m^{i2}} \; Var\left[F_j^i\right] + Var\left[FI_{j-1}^i\right]$$

It is apparent that it will be necessary to compute

$$Var\left[V_j^i FI_j^i\right]$$

for the next time step and that will be calculated when FI_j^i is found. The displaced mean and variance are

$$E\left[d_j^i\right] = E\left[d_{j-1}^i + V_j^i \Delta t\right]$$

$$= \text{deterministic calculations using mean values}$$

$$Var\left[d_j^i\right] = E\left[d_j^{i^2}\right] - E\left[d_j^i\right]^2$$

$$E\left[d_j^{i^2}\right] = Var\left[d_{j-1}^i\right] + 2\Delta t Cov\left[{}_{j-1}^i V_{j-1}^i\right] - 2\frac{\Delta t^2}{m^i} Cov\; d_{j-1}^i FI_{j-1}^i$$

$$+ Var\left[V_j^i\right]$$

The equations for the random properties for the remaining parameters in the one-dimensional finite element algorithm have been developed and are available in the computer program found in the Appendix. The program computes the evolution of the multivariate random distribution of the response measures, including mean and variance for each parameter and covariance between parameters.

Now, if a normal or a lognormal distribution is selected to represent the form of the distribution, then the parameters exist to completely define the distribution at any time step.

Even though the response measures form a multivariate distribution and integration would require an n-fold integration scheme, the marginal probability density functions can be assumed to be normal or lognormal if the joint probability density is likewise normal or lognormal, so that some information can be obtained from the marginal density functions.

3. NUMERICAL EXAMPLE

A bar 5 m long, with properties as shown in Table 9, is subjected to a decaying exponential load of the form

$$F = Ae^{-\alpha t}$$

where

A = 4.15 MPa

α = 1000

t = time in seconds

The above problem has been solved using a finite element procedure with A as a random parameter with mean 4.15 and standard deviation 0.4. Response measures of the first two nodes are shown in Figures 19 through 22. A one-standard deviation bound on the mean is shown for each parameter. A second problem is solved which included random material properties as shown in Table 10. This solution is also shown on Figures 19 through 22.

TABLE 9. PROPERTIES OF BAR

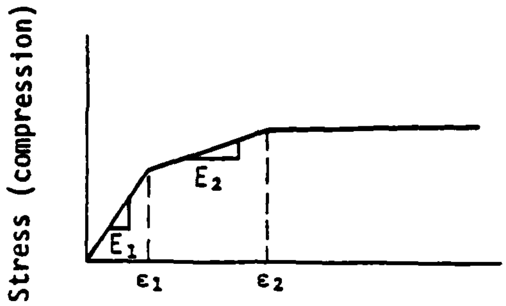
Area = 0.01 m^2

Density = 7840 kg/m^3

E_1 = 2 E+11 Pa ε_1 = -0.022

E_2 = 1 E+11 Pa ε_2 = -0.005

where E_1, E_2, ε_1, and ε_2 are defined by the following:



Strain (compression)

The model unloads on the initial modules, E.

TABLE 10. RANDOM MATERIAL PROPERTIES

Parameter	Mean, Pa	Standard deviation, Pa
E_1	2 E+11	2 E+10
E_2	1 E+11	1 E+10
ε_1	-0.002	0.0002
ε_2	-0.005	0.0001

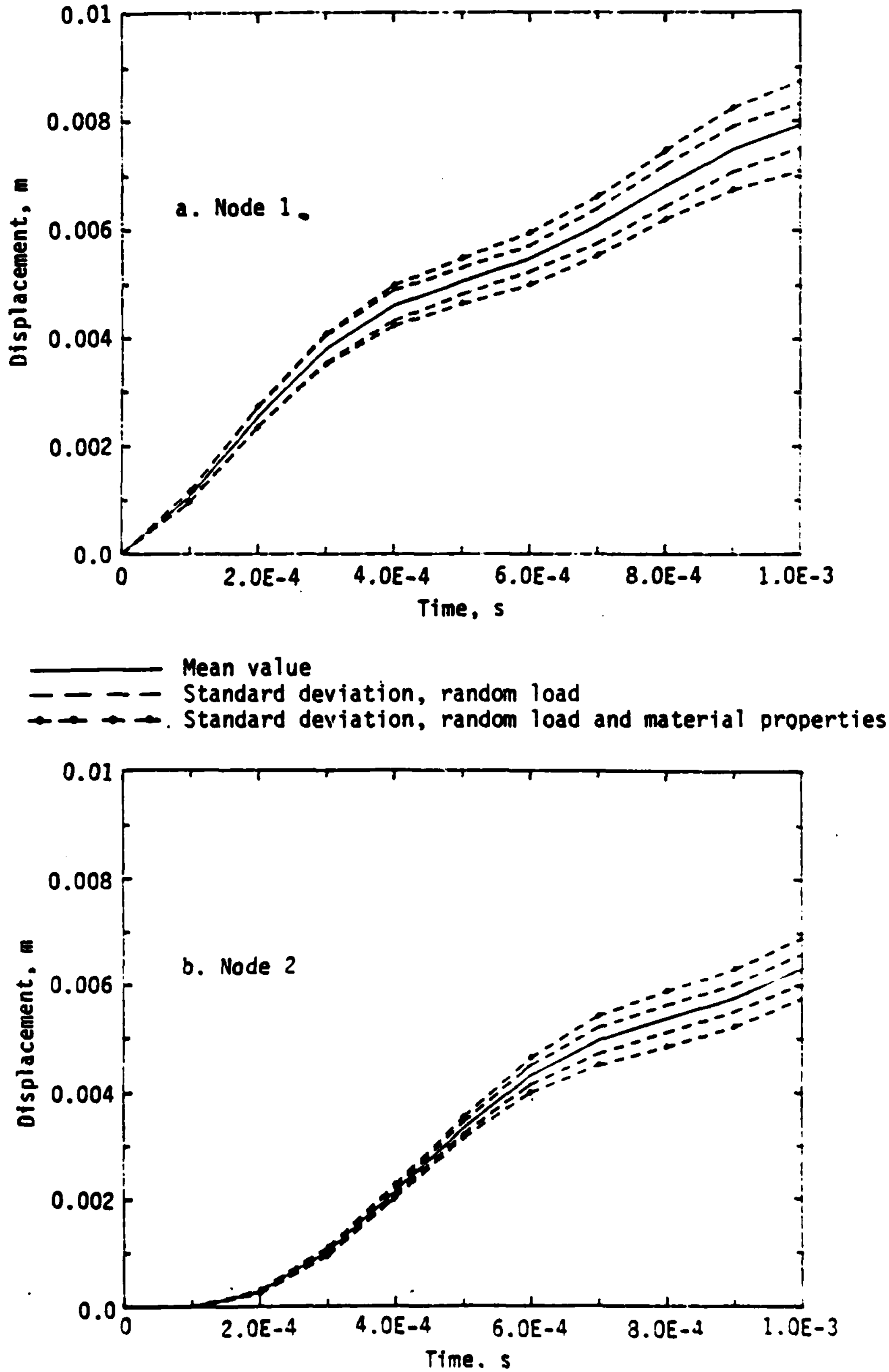


Figure 19. Random character of displacement, nodes 1 and 2.

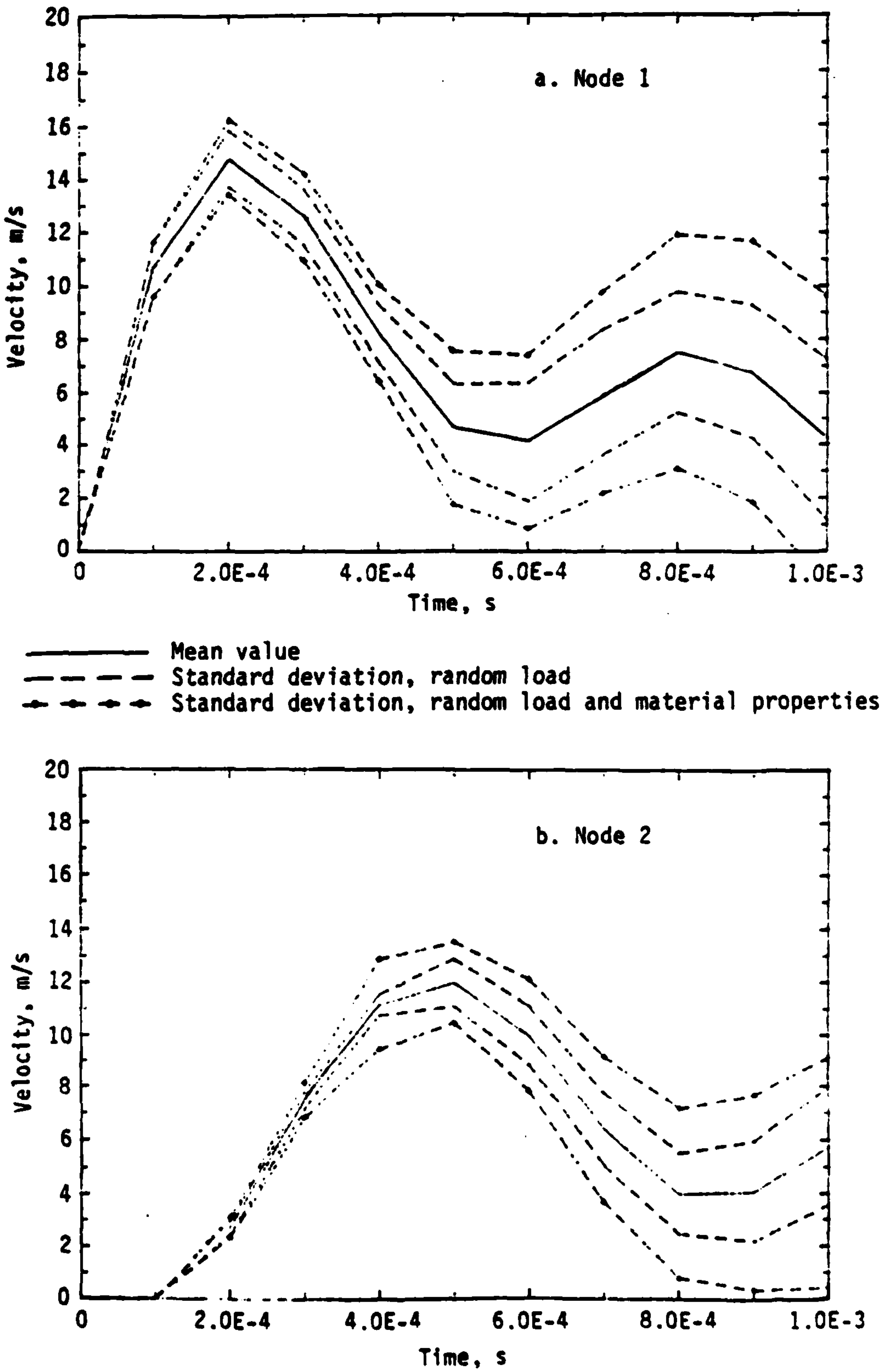


Figure 20. Random character of velocity, nodes 1 and 2.

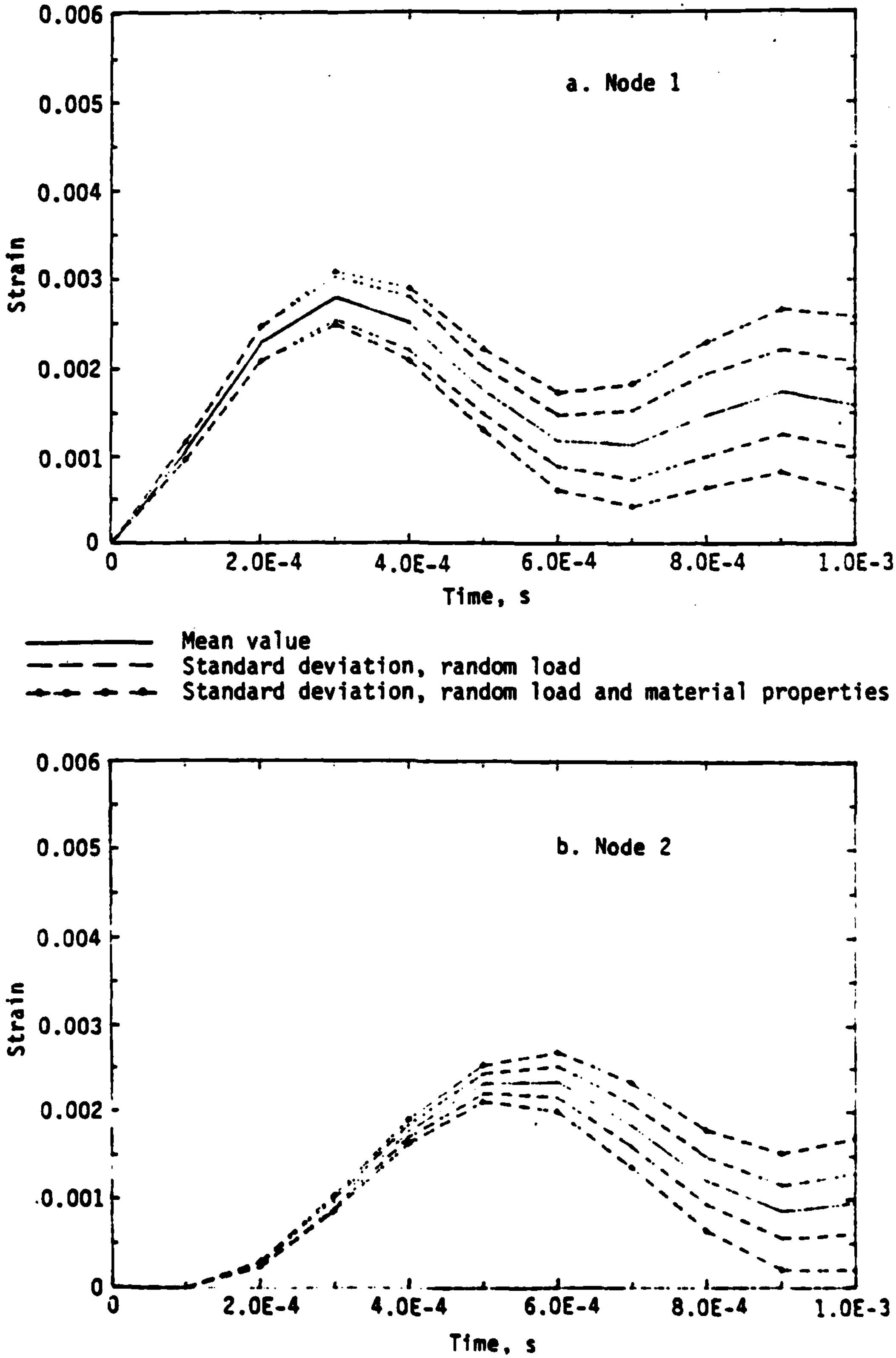


Figure 21. Random character of strain, nodes 1 and 2.

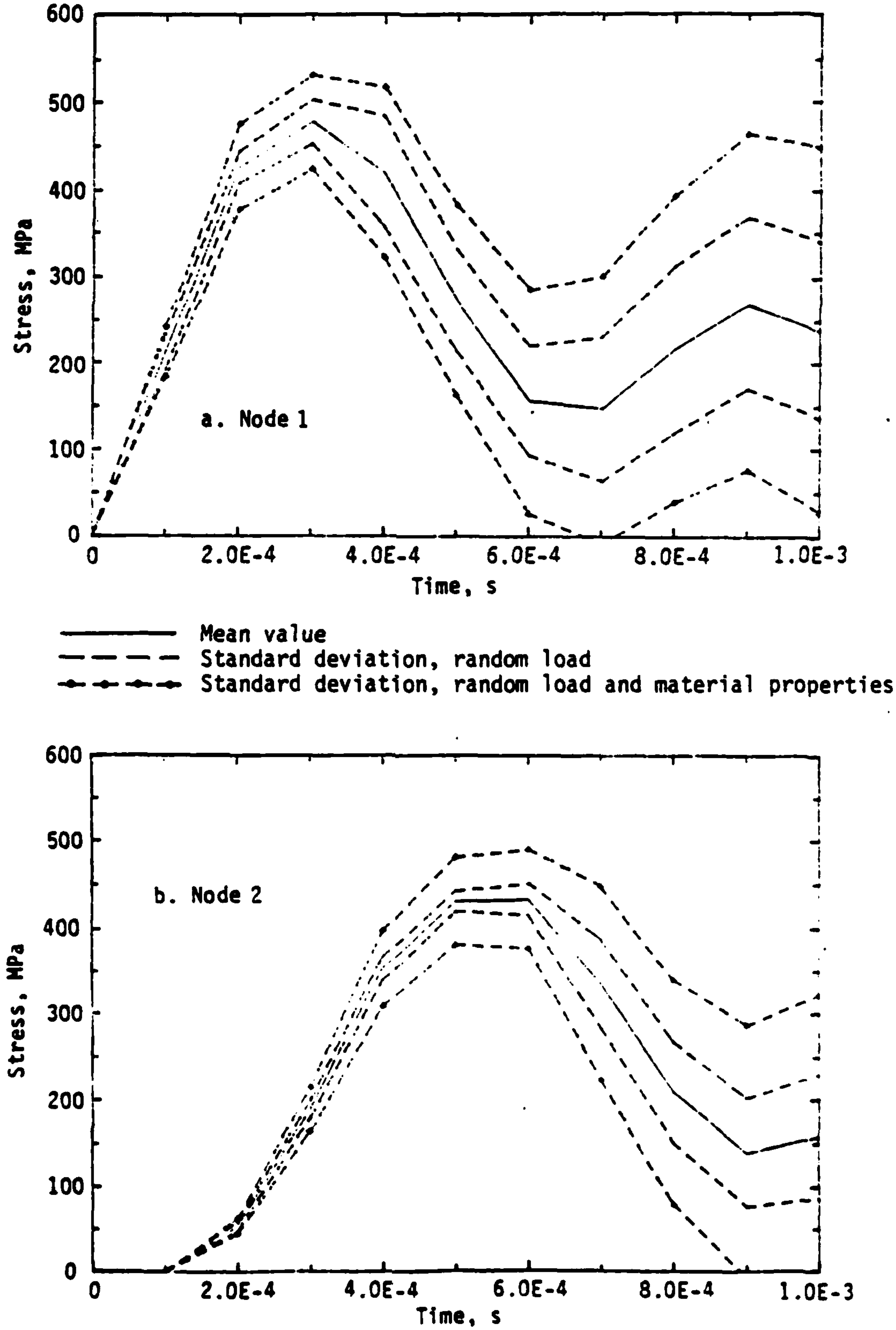


Figure 22. Random character of stress, nodes 1 and 2.

V. CONCLUSIONS AND RECOMMENDATIONS

1. CONCLUSIONS

Procedures have been developed to compute the probability of failure of structures subjected to blast and shock loads. The requirements are as follows:

a. The characteristics of the random load must be specified.

b. The structure system parameters that are considered random must be adequately specified.

c. Deterministic tools to compute the response must be available.

d. The random nature of the failure criteria must be characterized.

There are limitations to the presented approaches. For the approach presented for SDF systems, the computational storage requirement is quite large. Methods to reduce the requirements rely on narrow band response assumptions which are not correct for highly nonlinear systems. The discretization of response space introduces further numerical error and must be carefully selected with respect to time step and response measure history results.

The one approach presented for MDF representations is limited by the requirement to truncate the Taylor series at a reasonable order so as to limit the number of deterministic calculations that must be run to provide the numerical approximations of the partial differences. This truncation has been at the linear terms for the example problems, and is adequate as long as the distribution of the response about its mean is narrow enough so that the linear approximation is adequate. Otherwise higher order terms must be maintained. The linear terms can be adequate for nonlinear response if the peak range over which the linear approximation is used accounts for a significant portion of the peak response distribution. That is, for reasonable bounds on the peak response distribution, the peak response function must be linear in the random parameters.

The second MDF approach (finite element) would require extensive development for two-dimensional problems.

The approaches do not address model uncertainty. Model uncertainty should not be addressed with a random property approach and appropriate procedures are outside the scope of this report.

In practical application of the approaches developed in this investigation, the most severe limitation is an inadequate data base from which to establish random character of the loads, the structure system parameters, and the failure criteria. Testing is very expensive because of the nature of the loads; and reasonable tests provide some damage to the structure, making them difficult to repeat. Further definition of what can be correctly addressed as a random variable is needed.

2. EXTENSIONS

The most useful steps beyond this study include:

a. more thorough criteria for determining the order of truncation of the Taylor series. This would be developed mostly by extensive experience with a variety of problems.

b. The one-dimensional finite element model could be expanded to two-dimension. It may be necessary to approximate some of the relationships to keep the development to a manageable state. More experience with the one-dimensional model would be helpful.

c. The distribution of response measures could be used to validate whether a deterministic model could produce random response for random inputs in the same way as it does for the real system. For example, given a reasonable random input loading, both a mathematical model and a test could be conducted for concrete specimens. The distribution of the strain from both could be compared to validate the mathematical model. Because of the complex stress/strain paths that are investigated, normalization of data along path segments would probably be required.

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